

Non-Gaussian Limiting Laws for the Entries of Regular Functions of the Wigner Matrices

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Abstract

This paper is a continuation of our paper [8] in which we proved the Central Limit Theorem for the matrix elements of differential functions of the real symmetric random Gaussian matrices (GOE). Here we consider the real symmetric random Wigner matrices having independent (modulo symmetry conditions) but not necessarily Gaussian entries. We show that in this case the matrix elements of sufficiently smooth functions of these random matrices have in general another limiting law which coincides essentially with the probability law of matrix entries.

1 Introduction

We are interested in asymptotic properties of matrix elements $\varphi_{jk}(M)$, $j, k = 1, \dots, n$, $n \rightarrow \infty$, where φ is a smooth enough test-function, and M is the Wigner matrix. We define the Wigner real symmetric matrix as follows:

$$M_n = n^{-1/2}W_n, \quad W_n = \{W_{jk}^{(n)} \in \mathbb{R}, W_{jk}^{(n)} = W_{kj}^{(n)}\}_{j,k=1}^n, \quad (1.1)$$

where $\{W_{jk}^{(n)}\}_{1 \leq j \leq k \leq n}$ are independent random variables satisfying

$$\mathbf{E}\{W_{jk}^{(n)}\} = 0, \quad \mathbf{E}\{(W_{jk}^{(n)})^2\} = w^2(1 + \delta_{jk}). \quad (1.2)$$

The case of the Gaussian random variables obeying (1.2) corresponds to the Gaussian Orthogonal Ensemble (GOE) (see e.g. [9]):

$$\widehat{M}_n = n^{-1/2}\widehat{W}_n, \quad \widehat{W}_n = \{\widehat{W}_{jk} = \widehat{W}_{kj} \in \mathbb{R}, \widehat{W}_{jk} \in \mathcal{N}(0, w^2(1 + \delta_{jk}))\}_{j,k=1}^n. \quad (1.3)$$

We will assume in what follows additional conditions on distributions of $W_{jk}^{(n)}$, mostly in the form of existence of certain moments of $W_{jk}^{(n)}$, whose order will depend on the problem under study.

In our paper [8] we have considered matrix elements of functions of the GOE matrices and have proved the following facts.

Theorem 1.1 *Let \widehat{M}_n be the GOE matrix (1.3), and $\varphi_{1,2} : \mathbb{R} \rightarrow \mathbb{C}$ be bounded test functions with bounded derivative. Denote*

$$(\varphi_i(\widehat{M}_n))_{jj}^\circ = (\varphi_i(\widehat{M}_n))_{jj} - \mathbf{E}\{(\varphi_i(\widehat{M}_n))_{jj}\} \quad (1.4)$$

and

$$\mathbf{Cov}\{(\varphi_1(\widehat{M}_n))_{jj}, (\varphi_2(\widehat{M}_n))_{jj}\} = \mathbf{E}\{(\varphi_1(\widehat{M}_n))_{jj}(\varphi_2(\widehat{M}_n))_{jj}^\circ\}.$$

Then we have for any $j_n = 1, \dots, n$

$$\begin{aligned} & \lim_{n \rightarrow \infty} n \mathbf{Cov} \left\{ (\varphi_1(\widehat{M}_n))_{j_n j_n}, (\varphi_2(\widehat{M}_n))_{j_n j_n} \right\} \\ &= \int_{-2w}^{2w} \int_{-2w}^{2w} \Delta \varphi_1 \Delta \varphi_2 \rho_{sc}(\lambda_1) \rho_{sc}(\lambda_2) d\lambda_1 d\lambda_2, \end{aligned} \quad (1.5)$$

where

$$\Delta \varphi = \varphi(\lambda_1) - \varphi(\lambda_2), \quad (1.6)$$

ρ_{sc} is the density of the semicircle law

$$\rho_{sc}(\lambda) = (2\pi w^2)^{-1} (4w^2 - \lambda^2)_+^{1/2}, \quad (1.7)$$

and $x_+ = \max\{0, x\}$.

Theorem 1.2 *Let \widehat{M}_n be the GOE matrix (1.3), and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be bounded function with bounded derivative. Then for any $j_n = 1, \dots, n$ the random variable $\sqrt{n} \varphi^\circ(\widehat{M}_n)_{j_n j_n}$ converges in distribution to the Gaussian random variable with zero mean and the variance*

$$V_d^{GOE}[\varphi] = \int_{-2w}^{2w} \int_{-2w}^{2w} (\Delta \varphi)^2 \rho_{sc}(\lambda_1) \rho_{sc}(\lambda_2) d\lambda_1 d\lambda_2. \quad (1.8)$$

In the present paper we prove a counterpart of Theorems 1.1 and 1.2 for the Wigner matrices. In particular, we show in Theorem 3.2 below that in this case the r.h.s. of (1.5) has an additional term proportional to the fourth cumulant κ_4 of non-diagonal entries (see (2.18) for the definition). This result is in accordance with that for centered linear eigenvalue statistics

$$\mathcal{N}_n^\circ[\varphi] = \text{Tr } \varphi^\circ(M_n) = \sum_{j=1}^n \varphi_{jj}^\circ(M_n) \quad (1.9)$$

of the Wigner matrices (see Theorem 3.6 of [7]). On the other hand the individual matrix elements $\sqrt{n} \varphi_{jj}^\circ(M)$ do not satisfy in general the Central Limit Theorem (CLT). In Theorem 3.4 of this paper we find limiting probability law for $\sqrt{n} \varphi_{jj}^\circ(M)$ which is not Gaussian in general but rather that of the sum of the probability law of entries of M_n modulo a certain rescaling. To obtain the CLT, one has to impose an integral condition on the test function, i.e., the set of test functions for which we have the CLT has the codimension one.

Our results of [8] and of this paper can be viewed as analogs of the E. Borel theorem on the limiting probability law of entries of orthogonal matrices of size n as $n \rightarrow \infty$ (see e.g. [2]).

Convention: We will use letter C for an absolute constant that does not depend on j , k , and n , and may be distinct on different occasions.

2 Technical Means

To make the paper self-consistent, we present here several technical facts that will be often used below. For the proof of these facts see e.g. [4, 7]

We start from the generalized Fourier transform, in fact the $\pi/2$ rotated Laplace transform (see e.g. [15], Sections 1.8-9 for its definition).

Proposition 2.1 *Let $f : \mathbb{R}_+ \rightarrow \mathbb{C}$ be a locally Lipschitzian and such that for some $\delta > 0$*

$$\sup_{t \geq 0} e^{-\delta t} |f(t)| < \infty, \quad (2.1)$$

and let $\tilde{f}: \{z \in \mathbb{C} : \Im z < -\delta\} \rightarrow \mathbb{C}$ be its generalized Fourier transform

$$\tilde{f}(z) = i^{-1} \int_0^\infty e^{-izt} f(t) dt. \quad (2.2)$$

The inversion formula is given by

$$f(t) = \frac{i}{2\pi} \int_L e^{izt} \tilde{f}(z) dz, \quad t \geq 0, \quad (2.3)$$

where $L = (-\infty - i\varepsilon, \infty - i\varepsilon)$, $\varepsilon > \delta$, and the principal value of the integral at infinity is used.

Denote for the moment the correspondence between functions and their generalized Fourier transforms as $f \leftrightarrow \tilde{f}$. Then we have:

- (i) $\int_0^t f(\tau) d\tau \leftrightarrow (iz)^{-1} \tilde{f}(z);$
- (ii) $\int_0^t f_1(t-\tau) f_2(\tau) d\tau := (f_1 * f_2)(t) \leftrightarrow i \tilde{f}_1(z) \tilde{f}_2(z);$
- (iii) if P , Q , and R are differentiable, and $R(0) = 0$, then the equation

$$P(t) + \int_0^t dt_1 \int_0^{t_1} Q(t_1 - t_2) P(t_2) dt_2 = R(t), \quad t \geq 0, \quad (2.4)$$

has a unique differentiable solution

$$P(t) = - \int_0^t T(t - t_1) R'(t_1) dt_1, \quad (2.5)$$

where

$$T \leftrightarrow (z + \tilde{Q})^{-1} \quad (2.6)$$

provided by

$$z + \tilde{Q}(z) \neq 0, \quad \Im z < 0. \quad (2.7)$$

The next proposition presents simple facts of linear algebra

Proposition 2.2 *Let M and M' be $n \times n$ matrices and $t \in \mathbb{R}$. Then we have the following:*

- (i) the Duhamel formula

$$e^{(M+M')t} = e^{Mt} + \int_0^t e^{M(t-s)} M' e^{(M+M')s} ds. \quad (2.8)$$

- (ii) if for a real symmetric M

$$U(t) = e^{itM}, \quad t \in \mathbb{R}, \quad (2.9)$$

then $U(t)$ is a symmetric unitary matrix and

$$U(t_1)U(t_2) = U(t_1 + t_2), \quad \|U(t)\| = 1, \quad \sum_{j=1}^n |U_{jk}(t)|^2 = 1, \quad (2.10)$$

so that

$$|U_{jk}(t)| \leq 1, \quad \sum_{k=1}^n |U_{jk}(t)| \leq n^{1/2}, \quad \sum_{k=1}^n |U_{jk}(t_1)U_{j'k}(t_2)| \leq 1; \quad (2.11)$$

(iii)

$$D_{jk}U_{ab}(t) = i\beta_{jk}(U_{aj} * U_{bk} + U_{bj} * U_{ak})(t), \quad (2.12)$$

where

$$D_{jk} = \partial / \partial M_{jk}, \quad (2.13)$$

$$\beta_{jk} = (1 + \delta_{jk})^{-1} = 1 - \delta_{jk}/2, \quad (2.14)$$

the symbol " $*$ " is defined in Proposition 2.1 (ii), and

$$|D_{jk}^l U_{ab}(t)| \leq c_l |t|^l, \quad c_l = 2^l / l!. \quad (2.15)$$

Now a generalization of a property of the Gaussian random variable ξ of zero mean and of variance w^2 according to which if $\Phi : \mathbb{R} \rightarrow \mathbb{C}$ is a differentiable function with polynomially bounded derivative, then

$$\mathbf{E}\{\xi \Phi(\xi)\} = \mathbf{E}\{\xi^2\} \mathbf{E}\{\Phi'\}. \quad (2.16)$$

Formula (2.16) is a particular case of more general formula. To write it we recall some definitions. If a random variable ξ has a finite p th absolute moment, $p \geq 1$, then we have the expansions

$$f(t) := \mathbf{E}\{e^{it\xi}\} = \sum_{j=0}^p \frac{\mu_j}{j!} (it)^j + o(t^p),$$

and

$$l(t) := \log \mathbf{E}\{e^{it\xi}\} = \sum_{j=0}^p \frac{\kappa_j}{j!} (it)^j + o(t^p), \quad t \rightarrow 0, \quad (2.17)$$

where "log" denotes the principal branch of logarithm, the coefficients in the expansion of f are the moments $\{\mu_j\}$ of ξ , and the coefficients in the expansion of l are the cumulants $\{\kappa_j\}$ of ξ . For small j one easily expresses κ_j via $\mu_1, \mu_2, \dots, \mu_j$. In particular,

$$\begin{aligned} \kappa_1 &= \mu_1, \quad \kappa_2 = \mu_2 - \mu_1^2 = \mathbf{Var}\{\xi\}, \quad \kappa_3 = \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3, \\ \kappa_4 &= \mu_4 - 3\mu_2^2 - 4\mu_3\mu_1 + 12\mu_2\mu_1^2 - 6\mu_1^4, \dots \end{aligned} \quad (2.18)$$

In general

$$\kappa_j = \sum_{\lambda} c_{\lambda} \mu_{\lambda}, \quad (2.19)$$

where the sum is over all additive partitions λ of the set $\{1, \dots, j\}$, c_{λ} are known coefficients and $\mu_{\lambda} = \prod_{l \in \lambda} \mu_l$, see e.g. [14]. We have

Proposition 2.3 *Let ξ be a random variable such that $\mathbf{E}\{|\xi|^{p+2}\} < \infty$ for a certain non-negative integer p . Then for any function $\Phi : \mathbb{R} \rightarrow \mathbb{C}$ of the class C^{p+1} with bounded partial derivatives $\Phi^{(l)}$, $l = 1, \dots, p+1$, we have*

$$\mathbf{E}\{\xi \Phi(\xi)\} = \sum_{l=0}^p \frac{\kappa_{l+1}}{l!} \mathbf{E}\{\Phi^{(l)}(\xi)\} + \varepsilon_p, \quad (2.20)$$

where

$$|\varepsilon_p| \leq C_p \mathbf{E}\{|\xi|^{p+2}\} \sup_{t \in \mathbb{R}} |\Phi^{(p+1)}(t)|, \quad C_p \leq \frac{1 + (3 + 2p)^{p+2}}{(p+1)!}. \quad (2.21)$$

If the characteristic function $\mathbf{E}\{e^{it|\xi|}\}$ is whole, and $\Phi \in C^{\infty}$, then

$$\mathbf{E}\{\xi \Phi(\xi)\} = \sum_{l=0}^{\infty} \frac{\kappa_{l+1}}{l!} \mathbf{E}\{\Phi^{(l)}(\xi)\} \quad (2.22)$$

provided that for some $a > 0$

$$|\mathbf{E}\{\Phi^{(l)}(\xi)\}| \leq a^l, \quad (2.23)$$

and for some $R = ca$, $c > 1$, κ_l satisfy the condition:

$$\sum_{l=0}^{\infty} \frac{|\kappa_{l+1}| R^l}{l!} < \infty. \quad (2.24)$$

3 Covariance of Matrix Elements

We show first that if M is the Wigner matrix with uniformly bounded sixth moments of its entries, and the test-function φ is essentially of class \mathbf{C}^3 , then the variance of $\sqrt{n}\varphi_{jj}(M)$ is of the order $O(1)$ as $n \rightarrow \infty$. We have

Lemma 3.1 *Let $M = n^{-1/2}W$ be the real symmetric Wigner matrix (1.1) – (1.2). Assume that:*

(i) *the third moments of its entries do not depend on j , k , and n :*

$$\mu_3 = \mathbf{E}\{(W_{jk}^{(n)})^3\}; \quad (3.1)$$

(ii) *the sixth moments are uniformly bounded:*

$$w_6 := \sup_{n \in \mathbb{N}} \max_{1 \leq j, k \leq n} \mathbf{E}\{(W_{jk}^{(n)})^6\} < \infty. \quad (3.2)$$

Then for any test-function $\varphi : \mathbb{R} \rightarrow \mathbb{C}$, whose Fourier transform

$$\widehat{\varphi}(t) = \frac{1}{2\pi} \int e^{-it\lambda} \varphi(\lambda) d\lambda \quad (3.3)$$

satisfies the condition

$$\int (1 + |t|)^3 |\widehat{\varphi}(t)| dt < \infty, \quad (3.4)$$

we have the bound

$$\begin{aligned} \mathbf{Var}\{\sqrt{n}\varphi_{jj}(M)\} &= \mathbf{E}\{|\sqrt{n}\varphi_{jj}^\circ(M)|^2\} \\ &\leq C \left(\int (1 + |t|)^3 |\widehat{\varphi}(t)| dt \right)^2. \end{aligned} \quad (3.5)$$

Proof. It follows from the Fourier inversion formula

$$\varphi(\lambda) = \int e^{i\lambda t} \widehat{\varphi}(t) dt \quad (3.6)$$

and the spectral theorem for symmetric matrices that

$$\varphi_{jj}^\circ(M) = \int \widehat{\varphi}(t) U_{jj}^\circ(t) dt, \quad (3.7)$$

where U is defined in (2.9). This and the Schwarz inequality yield

$$\begin{aligned} \mathbf{Var}\{\varphi_{jj}(M)\} &= \int \int \mathbf{E}\{U_{jj}(t_1) U_{jj}^\circ(t_2)\} \widehat{\varphi}(t_1) \widehat{\varphi}(t_2) dt_1 dt_2 \\ &\leq \left(\int \mathbf{Var}^{1/2}\{U_{jj}(t)\} |\widehat{\varphi}(t)| dt \right)^2. \end{aligned} \quad (3.8)$$

Now (3.5) follows from the estimate

$$\mathbf{Var}\{U_{jj}(t)\} \leq C(1 + |t|)^6/n \quad (3.9)$$

proved in Lemma 3.8 below (see Appendix, (3.106)), and condition (3.4). ■

Theorem 3.2 Let $M = n^{-1/2}W$ be the real symmetric Wigner matrix (1.1) – (1.2). Assume that the third and fourth moments do not depend on j, k , and n :

$$\mu_3 = \mathbf{E}\{(W_{jk}^{(n)})^3\}, \quad \mu_4 = \mathbf{E}\{(W_{jk}^{(n)})^4\}, \quad (3.10)$$

and the sixth absolute moments are uniformly bounded (see (3.2)). Let $\varphi_{1,2} : \mathbb{R} \rightarrow \mathbb{C}$ be the test-functions, whose Fourier transforms $\widehat{\varphi}_{1,2}$ (3.3) satisfy (3.4). Then we have for any $j = j_n \in [1, n]$:

$$\begin{aligned} \lim_{n \rightarrow \infty} n \mathbf{Cov}\{(\varphi_1(M))_{jj}, (\varphi_2(M))_{jj}\} &= \int_{-2w}^{2w} \int_{-2w}^{2w} \Delta\varphi_1 \Delta\varphi_2 \rho_{sc}(\lambda_1) \rho_{sc}(\lambda_2) d\lambda_1 d\lambda_2 \\ &+ \frac{\kappa_4}{w^8} \prod_{i=1}^2 \int_{-2w}^{2w} \varphi_i(\lambda) (w^2 - \lambda^2) \rho_{sc}(\lambda) d\lambda, \end{aligned} \quad (3.11)$$

where $\Delta\varphi$ is defined in (1.6), and

$$\kappa_4 = \mu_4 - 3w^4 \quad (3.12)$$

is the fourth cumulant of the off-diagonal entries (see (2.18)).

In particular,

$$\begin{aligned} V_d^W[\varphi] &:= \lim_{n \rightarrow \infty} \mathbf{Var}\{\sqrt{n}\varphi_{jj}(M)\} \\ &= V_d^{GOE}[\varphi] + \frac{\kappa_4}{w^8} \left| \int_{-2w}^{2w} \varphi(\mu) (w^2 - \mu^2) \rho_{sc}(\mu) d\mu \right|^2 \end{aligned} \quad (3.13)$$

with $V_d^{GOE}[\varphi]$ of (1.8).

Remark 3.3 (i). If φ is odd, then $V_d^W[\varphi] = V_d^{GOE}[\varphi]$.

(ii). Note that we choose here the Wigner matrix so that its first two moments matches the first two moments of the GOE matrix (see (1.2)). This fact allows to use known properties of GOE and lies at the basis of interpolation procedure widely used in the proof of Lemma 3.8 below. In fact this condition is pure technical one, and we can replace condition (1.2) with more general one and consider Wigner matrix $\widetilde{M} = n^{-1/2}\widetilde{W}$, satisfying

$$\begin{aligned} \mathbf{E}\{\widetilde{W}_{jk}^{(n)}\} &= 0, \quad 1 \leq j \leq k \leq n, \\ \mathbf{E}\{(\widetilde{W}_{jk}^{(n)})^2\} &= w^2, \quad j \neq k, \quad \mathbf{E}\{(\widetilde{W}_{jj}^{(n)})^2\} = w_2 w^2, \quad w_2 > 0. \end{aligned} \quad (3.14)$$

In this case there arise additional terms in (3.11) and (3.13) proportional to $w_2 - 2$. In particular, we have for the corresponding limiting variance

$$V_d^{\widetilde{W}}[\varphi] = V_d^W[\varphi] + (w_2 - 2)w^{-2} \left| \int_{-2w}^{2w} \varphi(\mu) \mu \rho_{sc}(\mu) d\mu \right|^2, \quad (3.15)$$

where $V_d^W[\varphi]$ is given by (3.13).

Proof. Let us write the covariation in the form (cf (3.8))

$$n \mathbf{Cov}\{(\varphi_1(M))_{jj}, (\varphi_2(M))_{jj}\} = \int \int C_n(t_1, t_2) \widehat{\varphi}_1(t_1) \widehat{\varphi}_2(t_2) dt_1 dt_2, \quad (3.16)$$

where

$$C_n(t_1, t_2) = n \mathbf{E}\{U_{jj}(t_1) U_{jj}^\circ(t_2)\},$$

and U is defined in (2.9). It follows from the Schwarz inequality and (3.9) that

$$C_n(t_1, t_2) \leq C(1 + |t_1|)^3(1 + |t_2|)^3.$$

This, (3.4), and (3.16) imply that it suffices to show that there are converging subsequences $\{C_{n_i}\}$ and function Cov such that we have for any converging subsequence $\{C_{n_i}\}$ and any $T > 0$

$$\lim_{i \rightarrow \infty} C_{n_i}(t_1, t_2) = Cov(t_1, t_2) \quad (3.17)$$

uniformly on the square $S_T = \{(t_1, t_2) \in \mathbb{R} : |t_1| \leq T, |t_2| \leq T\}$, and that plugging Cov in the r.h.s. of (3.16) we get the r.h.s. of (3.11).

Show first that the derivatives $\partial C_n / \partial t_i$, $i = 1, 2$, are bounded on S_T uniformly in n . We have

$$\frac{\partial C_n}{\partial t_1} = in \mathbf{E}\{(MU)_{jj}(t_1)U_{jj}^\circ(t_2)\} = i\sqrt{n} \sum_{k=1}^n \mathbf{E}\{W_{jk}^{(n)} \Phi_{jk}(t_1, t_2)\}, \quad (3.18)$$

where

$$\Phi_{jk}(t_1, t_2) = U_{jk}(t_1)U_{jj}^\circ(t_2). \quad (3.19)$$

A simple algebra based on (2.15) allows to obtain

$$|D_{jk}^l \Phi_{jk}(t_1, t_2)| = O(1), \quad n \rightarrow \infty, \quad (3.20)$$

uniformly in $(t_1, t_2) \in S_T$. Now applying differentiation formula (2.20) with $\Phi = \Phi_{jk}$ and $p = 2$ to every term of the r.h.s. of (3.18), we get

$$\frac{\partial C_n}{\partial t_1} = iw^2 \sum_{k=1}^n \beta_{jk}^{-1} \mathbf{E}\{D_{jk} \Phi_{jk}(t_1, t_2)\} + \frac{i\mu_3}{\sqrt{n}} \sum_{k=1}^n \mathbf{E}\{D_{jk}^2 \Phi_{jk}(t_1, t_2)\} + \varepsilon_2(t_1, t_2), \quad (3.21)$$

where in view of (2.21) and (3.20)

$$|\varepsilon_2(t_1, t_2)| \leq \frac{C_2 \mu_4}{n} \sum_{k=1}^n \sup_{M \in \mathcal{S}_n} |D_{jk}^3 \Phi_{jk}(t_1, t_2)| = O(1), \quad n \rightarrow \infty. \quad (3.22)$$

Here \mathcal{S}_n is the set of $n \times n$ real symmetric matrices. Now it follows from (2.12) and (3.19) that every term of $D_{jk}^2 \Phi_{jk}$ contains U_{jk} (see (3.35)). Taking into account that $\sum_{k=1}^n |U_{jk}| \leq n^{1/2}$ (see (2.11)), we see that the second term on the r.h.s. of (3.21) is of the order $O(1)$, $n \rightarrow \infty$ uniformly in $(t_1, t_2) \in S_T$. At last, using (2.12) we get for the first term on the r.h.s. of (3.21):

$$\begin{aligned} T_1^{(n)}(t_1, t_2) &:= w^2 \sum_{k=1}^n \beta_{jk}^{-1} \mathbf{E}\{D_{jk} \Phi_{jk}(t_1, t_2)\} \\ &= iw^2 \sum_{k=1}^n \mathbf{E}\{(U_{jj} * U_{kk} + U_{jk} * U_{jk})(t_1)U_{jj}^\circ(t_2) + 2U_{jk}(t_1)(U_{jj} * U_{jk})(t_2)\} \\ &= iw^2 \mathbf{E}\{[(U_{jj} * nv_n)(t_1) + t_1 U_{jj}(t_1)]U_{jj}^\circ(t_2) + 2 \int_0^{t_2} U_{jj}(t_1 + t_4)U_{jj}(t_2 - t_4)dt_4\}, \end{aligned}$$

where we denote

$$v_n(t) := n^{-1} \text{Tr } U(t) = n^{-1} \sum_{k=1}^n U_{kk}(t), \quad |v_n(t)| \leq 1. \quad (3.23)$$

Since

$$n \mathbf{E}\{v_n U_{jj} U_{jj}^\circ\} = \mathbf{E}\{v_n\} C_n + n \mathbf{E}\{v_n^\circ U_{jj} U_{jj}^\circ\}$$

and by (3.106) – (3.107) of Lemma 3.8 below

$$\mathbf{Var}\{U_{jj}\} = O(n^{-1}), \quad \mathbf{Var}\{v_n\} = O(n^{-2}), \quad n \rightarrow \infty, \quad (3.24)$$

uniformly in $(t_1, t_2) \in S_T$, then we finally have

$$\begin{aligned} T_1^{(n)}(t_1, t_2) = & iw^2 \int_0^{t_1} \mathbf{E}\{v_n(t_4)\} C_n(t_1 - t_4, t_2) dt_4 \\ & + 2iw^2 \int_0^{t_2} \mathbf{E}\{U_{jj}(t_1 + t_4)\} \mathbf{E}\{U_{jj}(t_2 - t_4)\} dt_4 + r_n(t_1, t_2), \end{aligned} \quad (3.25)$$

where

$$\begin{aligned} r_n(t_1, t_2) = & iw^2 \mathbf{E}\{[(U_{jj} * nv_n^\circ)(t_1) + t_1 U_{jj}(t_1)] U_{jj}^\circ(t_2)\} \\ & + 2 \int_0^{t_2} \mathbf{E}\{U_{jj}(t_1 + t_4) U_{jj}^\circ(t_2 - t_4)\} dt_4 = O(n^{-1/2}), \quad n \rightarrow \infty, \end{aligned} \quad (3.26)$$

and see that $T_1^{(n)}(t_1, t_2) = O(1)$, $n \rightarrow \infty$ uniformly in $(t_1, t_2) \in S_T$.

It follows from the above that the derivatives $\partial C_n / \partial t_i$, $i = 1, 2$, are bounded on S_T uniformly in n . Hence, there are converging subsequences $\{C_{n_i}\}$ and function Cov (depending on subsequence) such that (3.17) holds. Now we derive an integral equation for Cov showing that Cov is the same for every converging subsequences $\{C_{n_i}\}$ and leading via (3.16) to (3.11).

It follows from the Duhamel formula (2.8) that

$$C_n(t_1, t_2) = \int_0^{t_1} i n^{1/2} \sum_{k=1}^n \mathbf{E}\{W_{jk}^{(n)} \Phi_{jk}(t_3, t_2)\} dt_3$$

with $\Phi_{jk}(t_3, t_2)$ given by (3.19). We see that the integrand here coincides with the r.h.s. of (3.18). Hence, applying differentiation formula (2.20) with $p = 3$, we get (cf (3.21) – (3.22)):

$$C_n(t_1, t_2) = \int_0^{t_1} i \left[\sum_{l=1}^3 T_l^{(n)}(t_3, t_2) + \varepsilon_3(t_3, t_2) \right] dt_3, \quad (3.27)$$

where

$$T_l^{(n)}(t_3, t_2) = \frac{1}{l! n^{(l-1)/2}} \sum_{k=1}^n \kappa_{l+1, jk} \mathbf{E}\{D_{jk}^l \Phi_{jk}(t_3, t_2)\}, \quad l = 1, 2, 3, \quad (3.28)$$

$\kappa_{l, jk}$ is the l th cumulant of $W_{jk}^{(n)}$:

$$\kappa_{1, jk} = 0, \quad \kappa_{2, jk} = w^2 \beta_{jk}^{-1}, \quad \kappa_{3, jk} = \mu_3, \quad \kappa_{4, jk} = \kappa_4 - 9\delta_{jk} w^4, \quad (3.29)$$

(see (1.2), (2.18), and (3.12)), and in view of (2.21) and (3.20)

$$|\varepsilon_3(t_3, t_2)| \leq \frac{C_3 w_6^{5/6}}{n^{3/2}} \sum_{k=1}^n \sup_{M \in \mathcal{S}_n} |D_{jk}^4 \Phi_{jk}(t_3, t_2)| = O(n^{-1/2}), \quad n \rightarrow \infty. \quad (3.30)$$

We see that $T_1^{(n)}$ of (3.28) is given by (3.25) – (3.26), where by (3.105) – (3.106) of Lemma 3.8 below

$$\lim_{n \rightarrow \infty} \mathbf{E}\{v_n(t)\} = \lim_{n \rightarrow \infty} \mathbf{E}\{U_{jj}(t)\} = v(t) = \int e^{it\lambda} \rho_{sc}(\lambda) d\lambda. \quad (3.31)$$

This and (3.17) yield

$$\lim_{i \rightarrow \infty} T_1^{(n_i)}(t_3, t_2) = iw^2 \int_0^{t_3} v(t_4) Cov(t_3 - t_4, t_2) dt_4 + 2iw^2 \Phi(t_3, t_2), \quad (3.32)$$

where

$$\Phi(t_3, t_2) = i^{-1} \int_{-2w}^{2w} \int_{-2w}^{2w} e^{it_3\lambda} \frac{e^{it_2\lambda} - e^{it_2\mu}}{\lambda - \mu} \rho_{sc}(\lambda) \rho_{sc}(\mu) d\lambda d\mu. \quad (3.33)$$

Consider now $T_2^{(n)}$ of (3.28), and show that

$$\lim_{n \rightarrow \infty} T_2^{(n)}(t_3, t_2) = 0. \quad (3.34)$$

We have by (2.12) and (3.28) with $l = 2$:

$$\begin{aligned} T_2^{(n)}(t_3, t_2) = & -\frac{\mu_3}{n^{1/2}} \sum_{k=1}^n \beta_{jk}^2 \mathbf{E} \{ (U_{jk} * U_{jk} * U_{jk} + 3U_{jj} * U_{jk} * U_{kk})(t_3) U_{jj}^\circ(t_2) \\ & + 2(U_{jk} * U_{jk} + U_{jj} * U_{kk})(t_3) (U_{jj} * U_{jk})(t_2) \\ & + U_{jk}(t_3) (3U_{jj} * U_{jk} * U_{jk} + U_{jj} * U_{jj} * U_{kk})(t_2) \}. \end{aligned} \quad (3.35)$$

It follows from (2.11) that the contribution of the terms containing $U_{jk}U_{jk}U_{jk}$ is of the order $O(n^{-1/2})$, $n \rightarrow \infty$. Besides, since $n^{-1/2} \sum_{k=1}^n |U_{kk}(t)U_{jk}(\tau)| \leq 1$, then by the Schwarz inequality and (3.9) the contribution of the terms containing $U_{jj}^\circ(t_2)$ is also of the order $O(n^{-1/2})$, $n \rightarrow \infty$. So we are left with

$$-\frac{\mu_3}{n^{1/2}} \sum_{k=1}^n \mathbf{E} \{ 2(U_{jj} * U_{kk})(t_3) (U_{jj} * U_{jk})(t_2) + U_{jk}(t_3) (U_{jj} * U_{jj} * U_{kk})(t_2) \}.$$

Here by (3.109) of Lemma 3.8 below

$$\mathbf{Var} \left\{ n^{-1/2} \sum_{k=1}^n U_{jk}(\tau_1) U_{kk}(\tau_2) \right\} = O(n^{-1/2}), \quad n \rightarrow \infty, \quad (3.36)$$

$$\lim_{n \rightarrow \infty} \mathbf{E} \left\{ n^{-1/2} \sum_{k=1}^n U_{jk}(\tau_1) U_{kk}(\tau_2) \right\} = 0, \quad (3.37)$$

so that

$$\lim_{n \rightarrow \infty} \mathbf{E} \left\{ n^{-1/2} \sum_{k=1}^n U_{jk}(\tau_1) U_{kk}(\tau_2) U_{jj}(\tau_3) U_{jj}(\tau_4) \right\} = 0,$$

and we get (3.34).

Consider now $T_3^{(n)}$ of (3.28). We have

$$T_3^{(n)}(t_3, t_2) = \frac{\kappa_4}{6n} \sum_{k=1}^n \mathbf{E} \{ D_{jk}^3(U_{jk}(t_3) U_{jj}^\circ(t_2)) \} + O(n^{-1}), \quad n \rightarrow \infty,$$

where we replaced $\kappa_{4,jk}$ of (3.29) with κ_4 of (3.12) with the error term of the order $O(n^{-1})$, $n \rightarrow \infty$. It follows now from (2.11) – (2.12) that the contribution to $T_3^{(n)}$ due to any term of

$$n^{-1} \sum_{k=1}^n D_{jk}^3(U_{jk}(t_3) U_{jj}^\circ(t_2))$$

containing at least one off-diagonal element U_{jk} is of the order $O(n^{-1/2})$, $n \rightarrow \infty$. Besides, we have by (2.15) and (3.9) that the term $\mathbf{E} \{ U_{jj}^\circ(t_2) n^{-1} \sum_{k=1}^n D_{jk}^3 U_{jk}(t_3) \}$ is of the order $O(n^{-1/2})$, $n \rightarrow \infty$,

too. Thus, we are left with terms, containing only diagonal non-centered elements of U . There is only one such term, it arises in the term $3D_{jk}U_{jk}(t_3)D_{jk}^2U_{jj}^\circ(t_2)$ of the sum above, and by (2.12) its contribution to $T_3^{(n)}$ is

$$-\frac{\kappa_4 i}{n} \sum_{k=1}^n \mathbf{E}\{(U_{jj} * U_{kk})(t_3)(U_{jj} * U_{jj} * U_{kk})(t_2)\}.$$

In view of (3.9), (3.106), and (3.108) we can replace here all U_{jj} and U_{kk} with v in the limit $n \rightarrow \infty$, so that we have

$$\lim_{n \rightarrow \infty} T_3^{(n)}(t_3, t_2) = -\kappa_4 i(v * v)(t_3)(v * v * v)(t_2). \quad (3.38)$$

Summarizing (3.27), (3.30), (3.32) – (3.34), and (3.38) we obtain the equation with respect to Cov of (3.17):

$$Cov(t_1, t_2) + w^2 \int_0^{t_1} dt_3 \int_0^{t_3} v(t_4) Cov(t_3 - t_4, t_2) dt_4 = A(t_1, t_2), \quad (3.39)$$

where

$$A(t_1, t_2) = -2w^2 \int_0^{t_1} \Phi(t_3, t_2) dt_3 + \kappa_4 (v * v * v)(t_2) \int_0^{t_1} (v * v)(t_3) dt_3 \quad (3.40)$$

and Φ is given by (3.33). To solve (3.39) we use the generalized Fourier transform with respect to t_1 (see Proposition 2.1). Note, that equation (3.39) is of the form (2.4), corresponding to $\delta = 0$ in (2.1), thus we can use formulas (2.5) – (2.6) to write its solution. Since

$$\tilde{v}(z) := -i \int_0^\infty e^{-itz} v(t) dt = (2w^2)^{-1} (\sqrt{z^2 - 4w^2} - z), \quad (3.41)$$

with the branch that is determined by the asymptotic $\sqrt{z^2 - 4w^2} = z + O(z^{-1})$, $z \rightarrow \infty$, and $(z + w^2 \tilde{v}(z))^{-1} = -\tilde{v}(z)$, then we have for T of (2.6):

$$T(t) = \frac{i}{2\pi} \int_L e^{itz} \frac{dz}{z + w^2 \tilde{v}(z)} = -v(t). \quad (3.42)$$

Hence, the unique differentiable solution of (3.39) is given by

$$Cov(t_1, t_2) = -2w^2 \int_0^{t_1} v(t_1 - t_3) \Phi(t_3, t_2) dt_3 + \kappa_4 \prod_{j=1}^2 (v * v * v)(t_j). \quad (3.43)$$

We have by (3.33), (3.105), and a little algebra:

$$\int_0^{t_1} v(t_1 - t_3) \Phi(t_3, t_2) dt_3 = - \int_{[-2w, 2w]^3} \frac{e^{i\lambda_1 t_1} - e^{i\lambda_2 t_1}}{\lambda_1 - \lambda_2} \frac{e^{i\lambda_2 t_2} - e^{i\lambda_3 t_2}}{\lambda_2 - \lambda_3} \prod_{j=1}^3 \rho_{sc}(\lambda_j) d\lambda_j,$$

and by (3.41)

$$\begin{aligned} (v * v * v)(t) &= \frac{i}{2\pi} \int_L e^{itz} (2w^2)^{-3} (\sqrt{z^2 - 4w^2} - z)^3 dz \\ &= w^{-4} \int_{-2w}^{2w} e^{it\lambda} (w^2 - \lambda^2) \rho_{sc}(\lambda) d\lambda. \end{aligned}$$

Hence, putting these expressions in (3.43), and then plugging the result in (3.16), we finally get:

$$\begin{aligned}
& \lim_{n \rightarrow \infty} n \mathbf{Cov}\{(\varphi_1(M))_{jj}, (\varphi_2(M))_{jj}\} \\
&= -2w^2 \int_{[-2w, 2w]^3} \frac{(\varphi_1(\lambda_1) - \varphi_1(\lambda_2))(\varphi_2(\lambda_2) - \varphi_2(\lambda_3))}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)} \prod_{j=1}^3 \rho_{sc}(\lambda_j) d\lambda_j \\
&+ \kappa_4 \prod_{j=1}^2 w^{-4} \int_{-2w}^{2w} \varphi_j(\lambda)(w^2 - \lambda^2) \rho_{sc}(\lambda) d\lambda.
\end{aligned} \tag{3.44}$$

Writing the numerator in the first integral as

$$\varphi_1(\lambda_1)\varphi_2(\lambda_2) - \varphi_1(\lambda_1)\varphi_2(\lambda_3) - \varphi_1(\lambda_2)\varphi_2(\lambda_2) + \varphi_1(\lambda_2)\varphi_2(\lambda_3), \tag{3.45}$$

we observe that there is at least one integration which does not involve φ 's. This and the relation

$$\int \frac{\rho_{sc}(\mu) d\mu}{\mu - \lambda} = -\lambda/2w^2$$

allow us to deduce (3.11) from (3.44). A simple way to perform the corresponding calculations is to write the r.h.s. of (3.44) as the limit as $\varepsilon \rightarrow 0$ of the same expression in which λ_3 is replaced by $\lambda_3 + i\varepsilon$. One can also use the Poincaré - Bertrand formula [10] to deal with double singular integrals, appearing after plugging (3.45) in (3.44). ■

3.1 Limit Theorem for Matrix Elements

Theorem 3.4 *Consider the real symmetric Wigner random matrix of the form*

$$M_n = n^{-1/2} W_n, \quad W_n = \{W_{jk} \in \mathbb{R}, W_{jk} = W_{kj} = (1 + \delta_{jk})^{1/2} V_{jk}\}_{j,k=1}^n, \tag{3.46}$$

where $\{V_{jk}\}_{1 \leq j \leq k < \infty}$ are i.i.d. random variables such that

$$\mathbf{E}\{V_{11}\} = 0, \quad \mathbf{E}\{V_{11}^2\} = w^2,$$

and the functions $\ln \mathbf{E}\{e^{itV_{11}}\}$ are entire.

Then for any $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ whose Fourier transform (3.3) satisfies (3.4) and for any $j = j_n \in [1, n]$ the random variable $\sqrt{n} \varphi_{j_n j_n}^\circ(M)$ converges in distribution as $n \rightarrow \infty$ to the random variable ξ having the characteristic function

$$\mathbf{E}\{e^{ix\xi}\} = \exp\{(-x^2 V_d^W[\varphi] + w^2 x^{*2})/2\} f(x^*), \tag{3.47}$$

where $f(x) = \mathbf{E}\{e^{ixV_{11}}\}$,

$$x^* = \frac{\sqrt{2}x}{w^2} \int_{-2w}^{2w} \varphi(\mu) \mu \rho_{sc}(\mu) d\mu, \tag{3.48}$$

ρ_{sc} is the density of the semicircle law (1.7), and $V_d^W[\varphi]$ is given by (3.13).

Remark 3.5 Condition $W_{jk} = (1 + \delta_{jk})^{1/2} V_{jk}$ is pure technical. In particular, it can be shown that in the case of matrix $\tilde{M} = n^{-1/2} V$, the Theorem 3.4 holds true with

$$x^* = \frac{x}{w^2} \int_{-2w}^{2w} \varphi(\mu) \mu \rho_{sc}(\mu) d\mu,$$

and

$$\mathbf{E}\{e^{ix\xi}\} = \exp\{(-x^2 V_d^W[\varphi] + 2w^2 x^{*2})/2\} f(x^*).$$

Proof. Note first that in view of (2.14) and (3.46) we can write

$$W_{jk} = \beta_{jk}^{-1/2} V_{jk}. \quad (3.49)$$

Besides, since $\ln \mathbf{E}\{e^{itV_{11}}\}$ is entire then we have

$$\sum_{l=1}^{\infty} \frac{x^l |\kappa_{l+1}|}{l!} < \infty, \quad \forall x > 0, \quad (3.50)$$

where κ_l is the l th cumulant of V_{11} . We also have

$$\ln f(x) = -w^2 x^2 / 2 + \sum_{l=3}^{\infty} \frac{\kappa_l (ix)^l}{l!} \quad (3.51)$$

(see (2.17)).

We consider the characteristic functions

$$Z_{jn}(x) = \mathbf{E} \left\{ e^{ix\sqrt{n}\varphi_{jj}^{\circ}(M)} \right\} \quad (3.52)$$

and prove that for any $x \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} Z_{jn}(x) = \mathbf{E}\{e^{ix\xi}\} =: Z_d(x), \quad (3.53)$$

i.e., $Z_d(x)$ is given by the r.h.s. of (3.47).

Assume first that the Fourier transform (3.3) of φ satisfies

$$\int |\widehat{\varphi}(t)| |t|^l dt < C_{\varphi} l! \quad \forall l \in \mathbb{N}, \quad (3.54)$$

in particular, φ is analytic in $|z| < 1$. Since $Z_{jn}(0) = 1$ and $Z_{jn}(x)$ is continuous, we can write the relation

$$Z_{jn}(x) = 1 + \int_0^x Z'_{jn}(y) dy, \quad x \in \mathbb{R}, \quad (3.55)$$

showing that it suffices to prove that the sequence $\{Z'_{jn}\}$ is uniformly bounded on any finite interval and that for any converging subsequences $\{Z_{jn_i}\}_{i \geq 1}$ and $\{Z'_{jn_i}\}_{i \geq 1}$ there exists $Z(x)$, such that

$$\lim_{i \rightarrow \infty} Z_{jn_i}(x) = Z(x), \quad (3.56)$$

and

$$\lim_{i \rightarrow \infty} Z'_{jn_i}(x) = Z(x) \left[-x V_d^W[\varphi] + \sum_{l=3}^{\infty} \frac{\kappa_l x^{l-1}}{(l-1)!} \left(\frac{i\sqrt{2}}{w^2} \int_{-2w}^{2w} \varphi(\mu) \mu \rho_{sc}(\mu) d\mu \right)^l \right]. \quad (3.57)$$

Indeed, if yes, then $Z(x)$ is a continuous function, satisfying for every $x \in \mathbb{R}$ the equation

$$\begin{aligned} Z(x) = 1 - \int_0^x Z(y) & \left[-y V_d^W[\varphi] \right. \\ & \left. + \sum_{l=3}^{\infty} \frac{\kappa_l y^{l-1}}{(l-1)!} \left(\frac{i\sqrt{2}}{w^2} \int_{-2w}^{2w} \varphi(\mu) \mu \rho_{sc}(\mu) d\mu \right)^l \right] dy, \end{aligned} \quad (3.58)$$

whose unique solution is the r.h.s. of (3.47).

We denote

$$e_{jn}(x) = e^{ix\sqrt{n}\varphi_{jj}^{\circ}(M)}, \quad (3.59)$$

and write according to (3.7) and (3.52)

$$Z'_{jn}(x) = i\mathbf{E} \left\{ \sqrt{n}\varphi_{jj}^\circ(M) e^{ix\sqrt{n}\varphi_{jj}^\circ(M)} \right\} = i \int \widehat{\varphi}(t) Y_{jn}(x, t) dt, \quad (3.60)$$

where

$$Y_{jn}(x, t) = \sqrt{n}\mathbf{E}\{U_{jj}(t)e_{jn}^\circ(x)\}, \quad (3.61)$$

and U is defined in (2.9). It follows from the Schwarz inequality and (3.9) that

$$|Y_{jn}(x, t)| \leq C(1 + |t|)^3. \quad (3.62)$$

This and (3.54) with $l = 2$ yield that the sequence Z'_{jn} is uniformly bounded. Hence, there is a convergent subsequence $Z_{jn'}$, and by the dominated convergence theorem to find its limit as $n \rightarrow \infty$ it suffices to find the pointwise limit of the corresponding subsequence $Y_{jn'}$.

Let us show now that sequences $\{\partial Y_{jn}/\partial x\}$ and $\{\partial Y_{jn}/\partial t\}$ are uniformly bounded in $(t, x) \in K \subset \mathbb{R}_+^2$, $n \in \mathbf{N}$, for any bounded K , so that the sequence $\{Y_{jn}\}$ is equicontinuous on any finite set of \mathbb{R}_+^2 , and contains convergent subsequences.

Since $\overline{Y_{jn}(x, t)} = Y_{jn}(-x, -t)$, we can confine ourselves to the half-plane $\mathbb{R}_+^2 = \{t \geq 0, x \in \mathbb{R}\}$, and from now on $t > 0$.

It follows from (3.7) that

$$\frac{\partial}{\partial x} Y_{jn}(x, t) = i \int \widehat{\varphi}(t_1) n \mathbf{E}\{U_{jj}^\circ(t_1) U_{jj}^\circ(t) e_{jn}(x)\} dt_1,$$

where by (3.9) and the Schwarz inequality

$$n |\mathbf{E}\{U_{jj}^\circ(t_1) U_{jj}^\circ(t) e_{jn}(x)\}| \leq n \mathbf{Var}^{1/2}\{U_{jj}(t_1)\} \mathbf{Var}^{1/2}\{U_{jj}(t)\} \leq C(1 + |t|)^3(1 + |t_1|)^3.$$

Hence, in view of (3.54) the sequence $\{\partial Y_{jn}/\partial x\}$ is uniformly bounded.

We have also

$$\frac{\partial}{\partial t} Y_{jn}(x, t) = i\sqrt{n}\mathbf{E}\{(MU)_{jj}(t)e_{jn}^\circ(x)\} = i \sum_{k=1}^n \mathbf{E}\{W_{jk}\Phi_{jk}(x, t)\}, \quad (3.63)$$

where

$$\Phi_{jk}(x, t) = U_{jk}(t)e_{jn}^\circ(x). \quad (3.64)$$

To transform the r.h.s. of (3.63) and show its boundedness, we apply an analog of integration by parts formula proposed in Lemma 2.3. Note that $D_{jk}^l \Phi_{jk} = O(n^{l/2})$ as $n \rightarrow \infty$, hence, there is no such finite $p \in \mathbf{N}$ that ε_p of (2.20) vanishes as $n \rightarrow \infty$, and so we need infinite version of "integration by parts formula" given by (2.22). We will apply (2.22) to every term of the r.h.s. of (3.63), and to do this we check first that $\Phi_{jk}(x, t)$ satisfies condition (2.23). Indeed, using the Leibnitz rule we obtain

$$D_{jk}^l \Phi_{jk}(x, t) = \sum_{m=0}^l \binom{l}{m} D_{jk}^{l-m} U_{jk}(t) D_{jk}^m e_{jn}^\circ(x), \quad (3.65)$$

where

$$D_{jk}^m e_{jn}(x) = D_{jk}^{m-1} (ix\sqrt{n}e_{jn}(x) D_{jk}\varphi_{jj}(M)), \quad (3.66)$$

(see (3.59)), so that

$$D_{jk}^m e_{jn}(x) = e_{jn}(x) \sum_{r=1}^m (ix\sqrt{n})^r \sum_{\substack{q = (q_1, \dots, q_r) : \\ q_1 + \dots + q_r = m}} C_{q,r} \prod_{s=1}^r D_{jk}^{q_s} \varphi_{jj}(M),$$

and

$$\sum_{q,r} C_{q,r} \leq 2^m.$$

Hence,

$$|D_{jk}^m e_{jn}(x)| \leq (2\sqrt{n}(1+|x|))^m \max_{1 \leq r \leq m, \sum_{s=1}^r q_s = m} \prod_{s=1}^r |D_{jk}^{q_s} \varphi_{jj}(M)|,$$

where we have in view of (2.15) and (3.54)

$$|D_{jk}^{q_s} \varphi_{jj}(M)| \leq \int |\widehat{\varphi}(\theta)| |D_{jk}^{q_s} U_{jj}(\theta)| d\theta \leq C_\varphi 2^{q_s}, \quad (3.67)$$

so that

$$|D_{jk}^m e_{jn}(x)| \leq (4C_\varphi \sqrt{n}(1+|x|))^m. \quad (3.68)$$

This, (2.15), and (3.65) yield

$$|D_{jk}^l \Phi_{jk}(x, t)| \leq (4C_\varphi \sqrt{n}(1+|x|+t))^l, \quad x \in \mathbb{R}, t > 0. \quad (3.69)$$

Thus, $\Phi_{jk}(x, t)$ satisfies condition (2.23). Applying (2.22) to every term of the r.h.s. of (3.63) and taking into account (3.49), we get:

$$\frac{\partial}{\partial t} Y_{jn}(x, t) = i \sum_{l=1}^{\infty} \frac{\kappa_{l+1}}{l!} S_l^{(n)}, \quad S_l^{(n)}(x, t) = \frac{1}{(\sqrt{n})^l} \sum_{k=1}^n \beta_{jk}^{-(l+1)/2} \mathbf{E}\{D_{jk}^l \Phi_{jk}(x, t)\}. \quad (3.70)$$

Let us show that this series converges uniformly in $(t, x) \in K$, $n \in \mathbb{N}$. In view of (3.50) it suffices to show that

$$|S_l^{(n)}| \leq (C_K)^l, \quad \forall (t, x) \in K, n \in \mathbb{N}, \quad (3.71)$$

where C_K is an absolute constant depending only on K . Since

$$S_l^{(n)}(x, t) = \frac{2^{(l+1)/2} - 1}{(\sqrt{n})^l} \mathbf{E}\{D_{jj}^l \Phi_{jj}(x, t)\} + \frac{1}{(\sqrt{n})^l} \sum_{k=1}^n \mathbf{E}\{D_{jk}^l \Phi_{jk}(x, t)\} \quad (3.72)$$

(see (2.14)), where in view of (3.69) the first term of the r.h.s. is bounded, it suffices to prove (3.71) for the second term of the r.h.s. of (3.72).

Using the Leibnitz rule, we write for $l \geq 2$

$$\begin{aligned} \frac{1}{(\sqrt{n})^l} \sum_{k=1}^n D_{jk}^l \Phi_{jk}(x, t) &= \frac{1}{(\sqrt{n})^l} \sum_{k=1}^n U_{jk} D_{jk}^l e_{jn} + \frac{l}{(\sqrt{n})^l} \sum_{k=1}^n D_{jk} U_{jk} D_{jk}^{l-1} e_{jn} \\ &+ \frac{1}{(\sqrt{n})^l} \sum_{k=1}^n \sum_{m=0}^{l-2} \binom{l}{m} D_{jk}^{l-m} U_{jk} D_{jk}^m e_{jn} =: a_{l1}^{(n)} + a_{l2}^{(n)} + a_{l3}^{(n)}, \end{aligned} \quad (3.73)$$

where (cf (3.65) – (3.69))

$$|a_{l3}^{(n)}| \leq (4C_\varphi(1+|x|+t))^l. \quad (3.74)$$

Applying (3.66) and then the Leibnitz rule again, we obtain for $a_{l1}^{(n)}$ of (3.73):

$$\begin{aligned} a_{l1}^{(n)} &= \frac{ix}{(\sqrt{n})^{l-1}} \sum_{k=1}^n U_{jk}(t) D_{jk} \varphi_{jj}(M) D_{jk}^{l-1} e_{jn} \\ &+ \frac{ix}{(\sqrt{n})^{l-1}} \sum_{k=1}^n U_{jk}(t) \sum_{m=0}^{l-2} \binom{l-1}{m} D_{jk}^{l-1-m} \varphi_{jj}(M) D_{jk}^m e_{jn}, \end{aligned} \quad (3.75)$$

where the sum over m is bounded by

$$(\sqrt{n})^{l-2}(4C_\varphi(1+|x|+t))^{l-1}$$

(cf (3.69) and (3.74)). Taking into account that $|\sum_{k=1}^n U_{jk}(t)| \leq n^{-1/2}$, we see that the second term of the r.h.s. of (3.75) is bounded by

$$(8C_\varphi(1+|x|+t))^l.$$

Besides, it follows from (3.7) and (2.12) that

$$D_{jk}\varphi_{jj}(M) = 2i\beta_{jk} \int \widehat{\varphi}(\theta) \int_0^\theta U_{jj}(\theta - \theta_1) U_{jk}(\theta_1) d\theta_1 d\theta, \quad (3.76)$$

$$|D_{jk}\varphi_{jj}(M)| \leq 2 \int_0^\infty |\widehat{\varphi}(\theta)| \int_0^\theta |U_{jk}(\theta_1)| d\theta_1 d\theta \leq 2C_\varphi. \quad (3.77)$$

This, (3.67) – (3.68), and (2.11) allow us to show that the first term of the r.h.s. of (3.75) is bounded by

$$\begin{aligned} 2|x|(4C_\varphi(1+|x|))^{l-1} \int_0^\infty |\widehat{\varphi}(\theta)| \int_0^\theta \sum_{k=1}^n |U_{jk}(t)| |U_{jk}(\theta_1)| d\theta_1 d\theta \\ \leq (4C_\varphi(1+|x|+|t|))^l. \end{aligned}$$

Hence,

$$|a_{l1}^{(n)}| \leq (4C_\varphi(1+|x|+t))^l. \quad (3.78)$$

Finally, since by (2.12) – (2.14)

$$D_{jk}U_{jk}(t) = i(U_{jj} * U_{kk} + U_{jk} * U_{jk})(t) - i\delta_{jk}(U_{jj} * U_{jj})(t),$$

we have for $a_{l2}^{(n)}$ of (3.73):

$$\begin{aligned} a_{l2}^{(n)} &= \frac{il}{(\sqrt{n})^l} \sum_{k=1}^n (U_{jj} * U_{kk})(t) D_{jk}^{l-1} e_{jn} + \frac{il}{(\sqrt{n})^l} \sum_{k=1}^n (U_{jk} * U_{jk})(t) D_{jk}^{l-1} e_{jn} \\ &\quad - \frac{il}{(\sqrt{n})^l} (U_{jj} * U_{jj})(t) D_{jj}^{l-1} e_{jn}, \end{aligned}$$

where the last two terms are bounded by $l(1+t)(4C_\varphi(1+|x|))^{l-1}$ in view of (3.68), (2.11), and the bound $|(U_{ab} * U_{cd})(t)| \leq |t|$. Besides, it follows from (3.66) and (3.77) that the first term is bounded by:

$$\begin{aligned} \frac{lt}{(\sqrt{n})^l} \sum_{k=1}^n |D_{jk}^{l-1} e_{jn}| &= \frac{lt}{(\sqrt{n})^{l-1}} \sum_{k=1}^n |D_{jk}\varphi_{jj} D_{jk}^{l-2} e_{jn} + \sum_{m=0}^{l-3} \binom{l-2}{m} D_{jk}^{l-2-m} \varphi_{jj} D_{jk}^m e_{jn}| \\ &\leq \frac{lt|x|}{(\sqrt{n})^{l-1}} \left[2 \int_0^\infty |\widehat{\varphi}(\theta)| \int_0^\theta \sum_{k=1}^n |U_{jk}(\theta_1)| d\theta_1 d\theta (4C_\varphi \sqrt{n}(1+|x|))^{l-2} \right. \\ &\quad \left. + (\sqrt{n})^{l-1} (4C_\varphi(1+|x|+|t|))^{(l-2)} \right] \leq (4C_\varphi(1+|x|+t))^l, \end{aligned}$$

so that

$$|a_{l2}^{(n)}| \leq (4C_\varphi(1+|x|+t))^l. \quad (3.79)$$

Now (3.71) with $l \geq 2$ follows from (3.72) – (3.74) and (3.78) – (3.79). Hence, the series in (3.63) converges uniformly in $(t, x) \in K \subset \mathbb{R}_+^2$ and $n \in \mathbb{N}$.

To prove the boundedness of the sequence $\{\partial Y_{jn}/\partial t\}$, it remains to make sure that $S_1^{(n)}$ is bounded. Applying (2.12) – (2.14) and (3.66), we obtain

$$\begin{aligned} S_1^{(n)}(x, t) &= n^{-1/2} \sum_{k=1}^n \beta_{jk}^{-1} \mathbf{E}\{D_{jk} \Phi_{jk}(x, t)\} \\ &= in^{-1/2} \mathbf{E}\{(n(U_{jj} * v_n)(t) + tU_{jj}(t))e_{jn}^\circ(x)\} \\ &\quad - 2x \int \widehat{\varphi}(\theta) \int_0^\theta \mathbf{E}\{U_{jj}(\theta - \theta_1)U_{jj}(t + \theta_1)e_{jn}(x)\} d\theta_1 d\theta \end{aligned} \quad (3.80)$$

where v_n is defined in (3.23). Writing

$$\begin{aligned} n^{1/2} \mathbf{E}\{U_{jj}(t_1)v_n(t_2)e_{jn}^\circ(x)\} \\ = \mathbf{E}\{v_n(t_2)\}Y_{jn}(x, t_1) + n^{1/2} \mathbf{E}\{U_{jj}(t_1)v_n^\circ(t_2)e_{jn}^\circ(x)\}, \end{aligned} \quad (3.81)$$

and taking into account bounds (3.62), (3.107), $|U_{jj}| \leq 1$, and $|e_{jn}^\circ(x)| \leq 2$, we conclude that the r.h.s. of (3.81) is bounded, and so does $S_1^{(n)}$. Hence, the sequence $\{\partial Y_{jn}/\partial t\}$ is uniformly bounded in $(t, x) \in K \subset \mathbb{R}_+^2$, $n \in \mathbb{N}$.

Now it follows from the above that the sequence $\{Y_{jn}\}$ is equicontinuous on any bounded set of \mathbb{R}^2 . Hence, for any converging subsequence $\{Z_{jn_i}\}$ (see (3.56)) there is a converging subsequence $\{Y_{jn'_i}\}$ and function Y (which obviously depends on $\{Z_{jn_i}\}$) such that

$$\lim_{n'_i \rightarrow \infty} Y_{jn'_i} = Y, \quad \lim_{n'_i \rightarrow \infty} Z_{jn'_i} = Z. \quad (3.82)$$

We will show now that Y satisfies certain integral equation leading through (3.60) to (3.58), hence, to (3.47). This will finish the proof of the theorem under condition (3.54).

Applying the Duhamel formula (2.8) and then (3.63) and (3.70), we obtain

$$Y_{jn}(x, t) = i\sqrt{n} \int_0^t \sum_{k=1}^n \mathbf{E}\{M_{jk} \Phi_{jk}(x, t_1)\} dt_1 = i \int_0^t \sum_{l=1}^\infty \frac{\kappa_{l+1}}{l!} S_l^{(n)}(x, t_1) dt_1, \quad (3.83)$$

where Φ_{jk} and $S_l^{(n)}$ are defined in (3.64) and (3.70), respectively. In view of the uniform convergence of the series, to make the limiting transition as $n \rightarrow \infty$ it suffices to find the limits

$$S_l = \lim_{n \rightarrow \infty} S_l^{(n)}$$

for every fixed $l \in \mathbb{R}$.

Let us start with $S_1^{(n)}$. It follows from (3.80) – (3.81) that (cf (3.25) – (3.26))

$$\begin{aligned} S_1^{(n)}(x, t_1) &= i \int_0^{t_1} \mathbf{E}\{v_n(t_1 - t_2)\} Y_{jn}(x, t_2) dt_2 \\ &\quad - 2x Z_{jn}(x) \int \widehat{\varphi}(\theta) \int_0^\theta \mathbf{E}\{U_{jj}(\theta - \theta_1)\} \mathbf{E}\{U_{jj}(t_1 + \theta_1)\} d\theta_1 d\theta + r_n(x, t_1), \end{aligned} \quad (3.84)$$

where

$$\begin{aligned} r_n(x, t) &= in^{-1/2} \mathbf{E}\{[n(U_{jj} * v_n^\circ)(t) + tU_{jj}(t)] e_{jn}^\circ(x)\} \\ &\quad - 2x \int \widehat{\varphi}(\theta) \int_0^\theta \mathbf{E}\{U_{jj}(\theta - \theta_1)[U_{jj}^\circ(t + \theta_1)Z_{jn}(x) + U_{jj}(t + \theta_1)e_{jn}^\circ(x)]\} d\theta_1 d\theta, \end{aligned}$$

and by (3.24) and boundedness of U_{ab} and e_{jn} we have

$$r_n = O(n^{-1/2}), \quad n \rightarrow \infty. \quad (3.85)$$

This, (3.31), and (3.82) imply

$$\lim_{n'_i \rightarrow \infty} S_1^{(n'_i)} = i \int_0^{t_1} v(t_1 - t_2) Y(x, t_1) dt_2 - 2xZ(x) \int \widehat{\varphi}(\theta) \Phi(t_1, \theta) d\theta, \quad (3.86)$$

where Φ is given by (3.33).

In general case we have for $S_l^{(n)}$, $l \geq 2$, of (3.70):

$$\begin{aligned} S_l^{(n)} &= \frac{1}{(\sqrt{n})^l} \sum_{k=1}^n \beta_{jk}^{-(l+1)/2} \mathbf{E} \left\{ U_{jk} D_{jk}^l e_{jn} + l D_{jk} U_{jk} D_{jk}^{l-1} e_{jn} \right. \\ &\quad \left. + l(l-1) D_{jk}^2 U_{jk} D_{jk}^{l-2} e_{jn} / 2 + (1 - \delta_{l2}) \sum_{m=0}^{l-3} \binom{l}{m} D_{jk}^{l-m} U_{jk} D_{jk}^m e_{jn} \right\} \\ &= S_{l1}^{(n)} + S_{l2}^{(n)} + S_{l3}^{(n)} + (1 - \delta_{l2}) S_{l4}^{(n)}, \end{aligned} \quad (3.87)$$

and we have (cf (3.74))

$$|S_{l4}^{(n)}| \leq (4C_\varphi(1 + |x| + |t|))^l n^{-1/2}, \quad l \geq 3. \quad (3.88)$$

Now we use the rule that in fact has been used several times before and which follows from (2.11) and the boundedness of e_{jn} of (3.59): the presence of a single factor U_{ak} in terms of the sum $\sum_{k=1}^n$ is equivalent to the presence of the factor $n^{-1/2}$, and the presence of two or more factors U_{ak}, U_{bk}, \dots is equivalent to the presence of the factor n^{-1} . It follows from (2.12) that all terms of $D_{jk}^2 U_{jk}$ contain U_{jk} , besides, we have

$$D_{jk}^{l-2} e_{jn} = O((\sqrt{n})^{l-2}), \quad n \rightarrow \infty \quad (3.89)$$

for $l > 2$, hence,

$$S_{l3}^{(n)} = O(n^{-1/2}), \quad n \rightarrow \infty. \quad (3.90)$$

We also have (see (3.75)):

$$\begin{aligned} S_{l1}^{(n)} &= \frac{ix}{(\sqrt{n})^{l-1}} \sum_{k=1}^n \beta_{jk}^{-(l+1)/2} \mathbf{E} \{ U_{jk}(t_1) [D_{jk} \varphi_{jj}(M) D_{jk}^{l-1} e_{jn} \\ &\quad + (l-1) D_{jk}^2 \varphi_{jj}(M) D_{jk}^{l-2} e_{jn}] \} + O(n^{-1/2}), \quad n \rightarrow \infty, \end{aligned}$$

where $D_{jk} \varphi_{jj}(M)$ is given by (3.76), and

$$D_{jk}^2 \varphi_{jj}(M) = -2i\beta_{jk}^2 \int \widehat{\varphi}(\theta) (3U_{jj} * U_{jk} * U_{jk} + U_{jj} * U_{jj} * U_{kk})(\theta) d\theta, \quad (3.91)$$

Hence, using again the above rule, we get

$$\begin{aligned} S_{l1}^{(n)} &= -2x \int \widehat{\varphi}(\theta) \frac{1}{(\sqrt{n})^{l-1}} \sum_{k=1}^n \beta_{jk}^{-(l-1)/2} \mathbf{E} \{ U_{jk}(t_1) (U_{jj} * U_{jk})(\theta) D_{jk}^{l-1} e_{jn} \\ &\quad + i(l-1) U_{jk}(t_1) (U_{jj} * U_{jj} * U_{kk})(\theta) D_{jk}^{l-2} e_{jn} \} d\theta \\ &\quad + O(n^{-1/2}), \quad n \rightarrow \infty. \end{aligned} \quad (3.92)$$

It follows from (3.36) – (3.37) that if $l = 2$ then the contribution of the second term of the r.h.s. of (3.92) vanishes as $n \rightarrow \infty$. To show this in the case $l \geq 3$ we use the evident relation

$$D_{jk}^m e_{jn}(x) = (ix\sqrt{n}D_{jk}\varphi_{jj}(M))^m e_{jn}(x) + O((\sqrt{n})^{m-1}), \quad n \rightarrow \infty, \quad (3.93)$$

with $D_{jk}\varphi_{jj}(M)$ given by (3.76), and conclude that the second term contains either $U_{jk}U_{jk} \cdot O((\sqrt{n})^{l-2})$ or $U_{jk} \cdot O((\sqrt{n})^{l-3})$, hence, we have in view of (2.11):

$$n^{-(l-1)/2} \sum_{k=1}^n \beta_{jk}^2 U_{jk}(t_1) (U_{jj} * U_{jj} * U_{kk})(\theta) D_{jk}^{l-2} e_{jn}(x) = O(n^{-1/2}), \quad n \rightarrow \infty.$$

The first term of the r.h.s. of (3.92) already contains $U_{jk}U_{jk}$. Thus its non-vanishing is due to the term $(ix\sqrt{n}D_{jk}\varphi_{jj}(M))^{l-1} e_{jn}(x)$ of $D_{jk}^{l-1} e_{jn}(x)$, and we get:

$$\begin{aligned} S_{l1}^{(n)} &= \sum_{k=1}^n \beta_{jk}^{-(l+1)/2} \mathbf{E} \{ U_{jk}(t_1) (ixD_{jk}\varphi_{jj}(M))^l e_{jn}(x) \} + O(n^{-1/2}) \\ &= \sum_{k=1}^n \mathbf{E} \left\{ e_{jn}(x) U_{jk}(t_1) \left(-2x \int \widehat{\varphi}(\theta) (U_{jk} * U_{jj})(\theta) d\theta \right)^l \right\} \\ &\quad + (2^{(1-l)/2} - 1) \mathbf{E} \left\{ e_{jn}(x) U_{jj}(t_1) \left(-2x \int \widehat{\varphi}(\theta) (U_{jj} * U_{jj})(\theta) d\theta \right)^l \right\} \\ &\quad + O(n^{-1/2}), \quad n \rightarrow \infty, \end{aligned}$$

where we took into account (3.76) and the equality $\beta_{jj}^{(l-1)/2} = 2^{(1-l)/2}$ (see (2.14)). Applying (3.104), (3.106), and (3.110) we get for $l \geq 2$:

$$\begin{aligned} &\mathbf{E} \left\{ e_{jn}(x) \sum_{k=1}^n U_{jk}(t_1) \prod_{m=1}^l U_{jk}(\theta_m) U_{jj}(\tau_m) \right\} \\ &= Z_{jn}(x) \bar{v}_{n2}(t_1, \theta_1, \dots, \theta_l) \prod_{m=1}^l \mathbf{E} \{ U_{jj}(\tau_m) \} + O(n^{-1/4}) \\ &= Z(x) v(t_1) \prod_{m=1}^l v(\theta_m) v(\tau_m) + o(1), \quad n_i \rightarrow \infty, \end{aligned}$$

and

$$\mathbf{E} \left\{ e_{jn}(x) \prod_{m=1}^{l'} U_{jj}(\tau_m) \right\} = Z(x) \prod_{m=1}^{l'} v(\tau_m) + o(1), \quad n_i \rightarrow \infty.$$

Besides, we have in view of (3.105)

$$(v * v)(\theta) = -\frac{i}{w^2} \int_{-2w}^{2w} e^{i\mu\theta} \mu \rho_{sc}(\mu) d\mu. \quad (3.94)$$

The above allows to write

$$\lim_{i \rightarrow \infty} S_{l1}^{(n_i)} = \sqrt{2} Z(x) v(t_1) \left(\frac{i\sqrt{2}x}{w^2} \int_{-2w}^{2w} \varphi(\mu) \mu \rho_{sc}(\mu) d\mu \right)^l. \quad (3.95)$$

It remains to analyze $S_{l2}^{(n)}$ of (3.87):

$$\begin{aligned}
S_{l2}^{(n)} &= \frac{l}{(\sqrt{n})^l} \sum_{k=1}^n \beta_{jk}^{-(l+1)/2} \mathbf{E}\{D_{jk}U_{jk}D_{jk}^{l-1}e_{jn}\} \\
&= -\frac{lx}{(\sqrt{n})^{l-1}} \sum_{k=1}^n \beta_{jk}^{-(l-1)/2} \mathbf{E}\{(U_{jj} * U_{kk} + U_{jk} * U_{jk})(t_1)D_{jk}^{l-2}(e_{jn}(x)D_{jk}\varphi_{jj}(M))\} \\
&= -\frac{lx}{(\sqrt{n})^{l-1}} \sum_{k=1}^n \mathbf{E}\left\{(U_{jj} * U_{kk})(t_1) \left(D_{jk}^{l-2}e_{jn}(x)D_{jk}\varphi_{jj}(M) \right. \right. \\
&\quad \left. \left. + (1 - \delta_{l2})(l-2)D_{jk}^{l-3}e_{jn}(x)D_{jk}^2\varphi_{jj}(M)\right)\right\} + O(n^{-1/2}), \quad n \rightarrow \infty,
\end{aligned}$$

where we used consequently (2.12), (3.66), and then the Leibnitz rule, (3.68), and (2.11). Treating the first term of the last expression analogously to the second term of (3.92), we see that it is of the order $O(n^{-1/2})$, $n \rightarrow \infty$. Hence, taking into account (3.91) and (2.11), we get $S_{22} = 0$, and for $l \geq 3$

$$\begin{aligned}
S_{l2}^{(n)} &= 2l(l-2)x \int \widehat{\varphi}(\theta) \frac{1}{(\sqrt{n})^{l-1}} \sum_{k=1}^n \mathbf{E}\left\{(U_{jj} * U_{kk})(t_1) \right. \\
&\quad \left. \times (U_{jj} * U_{jj} * U_{kk})(\theta)D_{jk}^{l-3}e_{jn}(x)\right\} d\theta + O(n^{-1/2}), \quad n \rightarrow \infty.
\end{aligned}$$

If $l > 3$, then in view of (3.93), $D_{jk}^{l-3}e_{jn}(x)$ gives either factor $U_{jk}O((\sqrt{n})^{l-3})$ or $O((\sqrt{n})^{l-4})$, that in both cases leads to

$$S_{l2}^{(n)} = O(n^{-1/2}), \quad n \rightarrow \infty, \quad l > 3. \quad (3.96)$$

If $l = 3$, then

$$\begin{aligned}
S_{32}^{(n)} &= 6x \int \widehat{\varphi}(\theta) \frac{1}{n} \sum_{k=1}^n \mathbf{E}\{(U_{jj} * U_{kk})(t_1)(U_{jj} * U_{jj} * U_{kk})(\theta)e_{jn}(x)\} d\theta \\
&\quad + O(n^{-1/2}), \quad n \rightarrow \infty,
\end{aligned}$$

and it follows from (3.102), (3.106), and (3.108) that

$$\begin{aligned}
&\mathbf{E}\{e_{jn}U_{jj}(\tau_1)U_{jj}(\tau_2)U_{jj}(\tau_3)n^{-1} \sum_{k=1}^n U_{kk}(\tau_4)U_{kk}(\tau_5)\} \\
&= Z_{jn}(x) \prod_{m=1}^3 \mathbf{E}\{U_{jj}(\tau_m)\} \mathbf{E}\{n^{-1} \sum_{k=1}^n U_{kk}(\tau_4)U_{kk}(\tau_5)\} + O(n^{-1/2}) \\
&= Z(z) \prod_{m=1}^5 v(\tau_m) + o(1), \quad n_i \rightarrow \infty.
\end{aligned}$$

Hence,

$$\lim_{n_i \rightarrow \infty} S_{32}^{(n_i)} = 6xZ(x)(v * v)(t_1) \int \widehat{\varphi}(\theta)(v * v * v)(\theta) d\theta.$$

This, (3.87) – (3.90), and (3.95) – (3.96) yield for $l \geq 2$:

$$\begin{aligned}
\lim_{n_i \rightarrow \infty} S_l^{(n_i)} &= Z(x)v(t_1) \left(\frac{i\sqrt{2}x}{w^2} \int_{-2w}^{2w} \varphi(\mu)\mu\rho_{sc}(\mu)d\mu \right)^l \\
&\quad + \delta_{3l}6xZ(x)(v * v)(t_1) \int \widehat{\varphi}(\theta)(v * v * v)(\theta) d\theta. \quad (3.97)
\end{aligned}$$

Summarizing (3.83), (3.86), and (3.97) we see that Y of (3.82) satisfies the equation

$$\begin{aligned} Y(x, t) + w^2 \int_0^t dt_1 \int_0^{t_1} v(t_1 - t_2) Y(x, t_2) dt_2 \\ = ixZ(x) \int \widehat{\varphi}(\theta) A(t, \theta) d\theta + i\sqrt{2}B(x)Z(x) \int_0^t v(t_1) dt_1, \end{aligned} \quad (3.98)$$

where A is defined in (3.40), and

$$B(x) = \sum_{l=2}^{\infty} \frac{\kappa_{l+1}}{l!} \left(\frac{i\sqrt{2}x}{w^2} \int_{-2w}^{2w} \varphi(\mu) \mu \rho_{sc}(\mu) d\mu \right)^l. \quad (3.99)$$

The kernel of this equation coincides with that of (3.39). Hence, the argument leading to (3.43) and based on using of generalized Fourier transform, yields

$$Y(x, t) = ixZ(x) \int \widehat{\varphi}(\theta) Cov(t, \theta) d\theta + i\sqrt{2}B(x)Z(x)(v * v)(t),$$

where Cov is given by (3.43). Plugging this expression in (3.60) and taking into account (3.94) and the equality (see (3.16))

$$\int \int \widehat{\varphi}(\theta) \widehat{\varphi}(t) Cov(t, \theta) dt d\theta = V_d^W[\varphi],$$

we finally get for Z of (3.82):

$$Z'(x) = Z(x) \left[-xV_d^W[\varphi] + B(x) \frac{i\sqrt{2}}{w^2} \int_{-2w}^{2w} \varphi(\mu) \mu \rho_{sc}(\mu) d\mu \right].$$

This yields (3.56) – (3.57), thus proves the theorem under condition (3.54).

The case of $\varphi \in E = \{\psi : \int (1+|t|)^3 |\widehat{\psi}(t)| dt < \infty\}$ can be obtained via a standard approximation procedure. Indeed, since the set $D = \{\varphi : \int |\widehat{\varphi}(t)| |t|^l dt < C_\varphi l!, \forall l \in \mathbb{N}\}$ is big enough (in particular, it contains functions $e^{-x^2} P_m(x)$, where $P_m(x)$ is a polynomial), then for any $\varphi \in E$ there exists a sequence $\{\varphi_k\} \subset D$, such that

$$\lim_{k \rightarrow \infty} \int_{-2w}^{2w} |\varphi(\lambda) - \varphi_k(\lambda)| d\lambda = 0. \quad (3.100)$$

Denote for the moment the characteristic functions of (3.52) and (3.53) as $Z_n[\varphi]$ and $Z[\varphi]$, to make explicit their dependence on φ . We have then for any $\varphi \in E$

$$\begin{aligned} |Z_n[\varphi] - Z[\varphi]| &\leq |Z_n[\varphi] - Z_n[\varphi_k]| + |Z_n[\varphi_k] - Z[\varphi_k]| + |Z[\varphi_k] - Z[\varphi]| \\ &:= T_{nk}^{(1)} + T_{nk}^{(2)} + T_{nk}^{(3)}. \end{aligned} \quad (3.101)$$

The second term of the r.h.s. vanishes after the limit $n \rightarrow \infty$ in view of the above proof, since $\varphi_k \in D$. For the first term we have from (3.52) and the Schwarz inequality that

$$|T_{nk}^{(1)}| \leq |x| (n \mathbf{Var}\{(\psi_k(M))_{jj}\})^{1/2}, \quad \psi_k = \varphi - \varphi_k,$$

and then Theorem 3.2 implies that

$$\limsup_{n \rightarrow \infty} |T_{nk}^{(1)}| \leq |x| (V_d^W[\psi_k])^{1/2}.$$

Since V_d^W of (3.13) is continuous with respect to the L^1 convergence, then in view of (3.100) $T_{nk}^{(1)}$ vanishes after the subsequent limits $n \rightarrow \infty$, $k \rightarrow \infty$.

At last, we have by (3.50) and the continuity of the r.h.s. of (3.47) with respect to the L^1 convergence, that the third term of (3.101) vanishes after the limit $k \rightarrow \infty$. Thus, we have proved the Central Limit Theorem under condition (3.4). ■

Corollary 3.6 *It follows from Theorem 3.4 that if φ is even, then the random variable $\sqrt{n}\varphi^\circ(M)_{jj}$ converges in distribution to the Gaussian random variable with zero mean and the variance $V_d^W[\varphi]$.*

Remark 3.7 Random matrix theory deals mostly with eigenvalues of large random matrices. However, the statistical properties of eigenvectors are also of considerable interests for a number of reasons, in particular in view of possible links with the problem of existence of absolutely continuous spectrum of the multidimensional Schrödinger operator with random potential (see e.g. [3, 5]). In the case of the Gaussian random matrices (GOE, null Wishart) the eigenvectors are rotationally invariant and according to recent works [1, 3, 6] the eigenvectors of the non-Gaussian random matrices (Wigner, sample covariance) are similar in several aspects to the eigenvectors of the Gaussian random matrices. On the other hand, the results of [8] and this papers imply that there are asymptotic properties of eigenvectors of the non-Gaussian random matrices which are different of those for the Gaussian random matrices.

3.2 Auxiliary results

Lemma 3.8 *Consider the unitary matrix $U(t) = e^{itM}$ of (2.9) – (2.10), where M is the Wigner matrix (1.1) – (1.2), define*

$$\begin{aligned} v_n(t) &= n^{-1} \sum_{k=1}^n U_{kk}(t), \\ v_n(t_1, t_2) &= n^{-1} \sum_{k=1}^n U_{kk}(t_1) U_{kk}(t_2), \end{aligned} \quad (3.102)$$

$$v_{n1}(t_1, t_2) = n^{-1/2} \sum_{k=1}^n U_{jk}(t_1) U_{kk}(t_2), \quad (3.103)$$

$$v_{n2}(\bar{t}) = \sum_{k=1}^n \prod_{m=1}^l U_{jk}(t_m), \quad l \geq 3, \quad (3.104)$$

where $\bar{t} = (t_1, \dots, t_l)$, and put $\bar{f} = \mathbf{E}\{f\}$. Then we have:

$$\lim_{n \rightarrow \infty} \bar{v}_n(t) = \int_{-2w}^{2w} e^{it\lambda} \rho_{sc}(\lambda) d\lambda =: v(t), \quad (3.105)$$

where ρ_{sc} is the density of the semicircle law (1.7), and under the conditions of Theorem 3.2

$$(i) \quad \mathbf{Var}\{U_{jj}(t)\} \leq C(1 + |t|)^6/n, \quad \lim_{n \rightarrow \infty} \bar{U}_{jj}(t) = v(t), \quad (3.106)$$

$$(ii) \quad \mathbf{Var}\{v_n(t)\} = O(n^{-2}), \quad n \rightarrow \infty, \quad (3.107)$$

$$(iii) \quad \mathbf{Var}\{v_n(t_1, t_2)\} = O(n^{-1/2}), \quad n \rightarrow \infty, \quad \lim_{n \rightarrow \infty} \bar{v}_n(t_1, t_2) = v(t_1)v(t_2), \quad (3.108)$$

$$(iv) \quad \mathbf{Var}\{v_{n1}(t_1, t_2)\} = O(n^{-1/2}), \quad n \rightarrow \infty, \quad \lim_{n \rightarrow \infty} \bar{v}_{n1}(t_1, t_2) = 0, \quad (3.109)$$

$$(v) \quad \mathbf{Var}\{v_{n2}(\bar{t})\} = O(n^{-1/2}), \quad n \rightarrow \infty, \quad \lim_{n \rightarrow \infty} \bar{v}_{n2}(\bar{t}) = \prod_{m=1}^l v(t_m), \quad (3.110)$$

where $O(n^\alpha)$ can depend on t, \bar{t} .

Remark 3.9 It can be shown that all statements of the lemma remain valid under conditions of Theorem 3.4

Proof. Statement (3.105) follows from the well known fact of random matrix theory (see e.g. [12] and references therein) according to which for any bounded and continuous φ

$$\lim_{n \rightarrow \infty} n^{-1} \mathbf{E}\{\text{Tr } \varphi(M)\} = \int_{-2w}^{2w} \varphi(\lambda) \rho_{sc}(\lambda) d\lambda,$$

where ρ_{sc} is the density of the semicircle law (1.7).

(i) Let $\widehat{M} = n^{-1/2} \widehat{W}$ be GOE matrix (1.3) independent of M , and

$$\widehat{U}(t) = e^{it\widehat{M}}. \quad (3.111)$$

We can write

$$V_n := \mathbf{Var}\{U_{jj}(t)\} = \mathbf{E}\{(U_{jj}(t) - \widehat{U}_{jj}(t))U_{jj}^\circ(-t)\}, \quad (3.112)$$

and then follow the interpolation procedure proposed in [8]. Namely, consider the "interpolating" random matrix (see [13, 8])

$$M(s) = s^{1/2}M + (1-s)^{1/2}\widehat{M}, \quad 0 \leq s \leq 1, \quad (3.113)$$

viewed as defined on the product of the probability spaces of matrices W and \widehat{W} . We denote again by $\mathbf{E}\{\dots\}$ the corresponding expectation in the product space. Since $M(1) = M$, $M(0) = \widehat{M}$, then putting

$$U(t, s) = e^{itM(s)}, \quad (3.114)$$

we obtain

$$\begin{aligned} U_{jj}(t) - \widehat{U}_{jj}(t) &= \int_0^1 \frac{\partial}{\partial s} U_{jj}(t, s) ds \\ &= \frac{i}{2} \int_0^1 \sum_{l,m=1}^n \left(\frac{1}{\sqrt{sn}} W_{lm}^{(n)} - \frac{1}{\sqrt{(1-s)n}} \widehat{W}_{lm} \right) (U_{jl} * U_{mj})(t, s) ds. \end{aligned} \quad (3.115)$$

Thus,

$$V_n = \frac{i}{2} \int_0^1 \left[\frac{1}{\sqrt{sn}} \sum_{l,m=1}^n \mathbf{E}\{W_{lm}^{(n)} \Phi_{lm}\} - \frac{1}{\sqrt{(1-s)n}} \sum_{l,m=1}^n \mathbf{E}\{\widehat{W}_{lm} \Phi_{lm}\} \right] ds,$$

where

$$\Phi_{lm} = (U_{jl} * U_{mj})(t, s) U_{jj}^\circ(-t). \quad (3.116)$$

A simple algebra based on (2.12) and (2.15) allows to obtain

$$|D_{lm}^q \Phi_{lm}| \leq C_q |t|^{q+1}, \quad (3.117)$$

with C_q depending only on $q \in \mathbb{N}$.

Now, applying differentiation formula (2.20) with $p = 4$ and $\Phi = \Phi_{lm}$ to every term of the first sum and differentiation formula (2.16) to every term of the second sum, we obtain (cf (3.27) – (3.30)):

$$V_n = \frac{i}{2} \int_0^1 \left[\sum_{l=2}^4 T_p^{(n)} + \varepsilon_4 \right] s^{-1/2} ds, \quad (3.118)$$

where

$$T_p^{(n)} = \frac{1}{p!n^{(p+1)/2}} \sum_{l,m=1}^n \kappa_{p+1,lm} \mathbf{E}\{D_{lm}^p \Phi_{lm}\}, \quad p = 2, 3, 4, \quad (3.119)$$

and by (2.21) and (3.117)

$$|\varepsilon_4| \leq \frac{C_4 w_6}{n^3} \sum_{l,m=1}^n \sup_{M \in \mathcal{S}_n} |D_{lm}^5 \Phi_{lm}| \leq C(1 + |t|)^6 n^{-1}. \quad (3.120)$$

Consider now $T_2^{(n)}$ and note that by (2.18) and (3.1) $\kappa_{3,lm} = \mu_3$, so that

$$\begin{aligned} T_2^{(n)} &= \frac{\mu_3}{2n^{3/2}} \sum_{l,m=1}^n \mathbf{E}\{D_{lm}^2 \Phi_{lm}\} \\ &= \frac{\mu_3}{2n^{3/2}} \sum_{l,m=1}^n \mathbf{E}\{U_{jj}^\circ(-t) D_{lm}^2 (U_{jl} * U_{mj})(t, s) \\ &\quad + (U_{jl} * U_{mj})(t, s) D_{lm}^2 U_{jj}(-t) \\ &\quad + 2D_{lm}(U_{jl} * U_{mj})(t, s) D_{lm} U_{jj}(-t)\} := \mu_3 [T_{21}^{(n)} + T_{22}^{(n)} + T_{23}^{(n)}]. \end{aligned} \quad (3.121)$$

It follows from (2.12) that

$$\begin{aligned} T_{21}^{(n)} &= -3\mu_3 s n^{-3/2} \sum_{l,m=1}^n \mathbf{E}\{U_{jj}^\circ(-t) (U_{jl} * U_{jm} * U_{lm} * U_{lm} \\ &\quad + 2U_{jl} * U_{jl} * U_{lm} * U_{mm} + U_{jl} * U_{jm} * U_{ll} * U_{mm})(t, s)\}. \end{aligned} \quad (3.122)$$

Here by (2.11)

$$\sum_{l,m=1}^n |U_{jl} U_{jm} U_{lm} U_{lm}| \leq 1, \quad \sum_{l,m=1}^n |U_{jl} U_{jl} U_{lm} U_{mm}| \leq n^{1/2}, \quad (3.123)$$

$$\sum_{l,m=1}^n |U_{jl} U_{jm} U_{ll} U_{mm}| \leq n. \quad (3.124)$$

Hence, applying the Schwarz inequality and taking into account that $(1 * 1 * 1 * 1)(t) = t^3/6$, we obtain

$$|T_{21}^{(n)}| \leq C(1 + |t|^3)(n^{-1/2} V_n^{1/2} + n^{-1}). \quad (3.125)$$

The terms $T_{22}^{(n)}$ and $T_{23}^{(n)}$ contain sums of two types

$$\sum_{l,m=1}^n U_{jl} U_{jl} U_{jl} U_{jm} U_{mm} = O(n^{1/2}), \quad \sum_{l,m=1}^n U_{jl} U_{jl} U_{jm} U_{jm} U_{lm} = O(1), \quad n \rightarrow \infty, \quad (3.126)$$

where the r.h.s. of both equalities follows from (2.11). Hence, $|T_{22}^{(n)} + T_{23}^{(n)}| \leq C(1 + |t|^3)n^{-1}$. This, (3.121), and (3.125) yield

$$|T_2^{(n)}| \leq C(1 + |t|^3)(n^{-1/2} V_n^{1/2} + n^{-1}). \quad (3.127)$$

Acting in the similar way and taking into account (2.19) and (3.2) implying $|\kappa_{p,jk}| \leq C$, $p = 4, 5$, we get analogous bounds for $T_3^{(n)}$ and $T_4^{(n)}$ of (3.118):

$$|T_l^{(n)}| \leq C(1 + |t|)^{l+1} n^{-1}, \quad l = 4, 5. \quad (3.128)$$

Note, that in the case of $T_3^{(n)}$ and $T_4^{(n)}$ the argument is even simpler because here we have factors n^{-2} and $n^{-5/2}$, respectively, instead of $n^{-3/2}$ of $T_2^{(n)}$.

Now it follows from (3.118) – (3.120) and (3.127) – (3.128) that $V_n^{1/2}$ satisfies the quadratic inequality:

$$V_n - C(1 + |t|)^3 n^{-1/2} V_n^{1/2} - C(1 + |t|)^6 n^{-1} \leq 0,$$

implying (3.9) and then (3.5).

To finish the proof of (i) it remains to show that

$$\lim_{n \rightarrow \infty} \mathbf{E}\{U_{jj}(t)\} = v(t). \quad (3.129)$$

In the GOE case we have by the orthogonal invariance of GOE probability measure and (3.105):

$$\mathbf{E}\{\widehat{U}_{jj}(t)\} = \mathbf{E}\{n^{-1} \text{Tr } \widehat{U}(t)\} \rightarrow v(t) \text{ as } n \rightarrow \infty. \quad (3.130)$$

Besides, it follows from (3.113), (3.115), (2.16), and (2.20) with $p = 3$, that

$$\mathbf{E}\{U_{jj}(t) - \widehat{U}_{jj}(t)\} = \frac{i}{2} \int_0^1 \left[\sum_{l=2}^3 T_p^{(n)} + \varepsilon_3 \right] s^{-1/2} ds,$$

where T_p are given by (3.119) with $\Phi_{lm} = (U_{jl} * U_{mj})(t, s)$ (cf (3.116)), and

$$|\varepsilon_3| \leq \frac{C_3 w_6^{5/6}}{n^{5/2}} \sum_{l,m=1}^n \sup_{M \in \mathcal{S}_n} |D_{lm}^4 \Phi_{lm}| \leq C(1 + |t|)^5 n^{-1/2}.$$

A similar but much simpler argument leading to (3.127) – (3.128) allows to conclude that $|T_p^{(n)}| \leq O(n^{-1/2})$, $n \rightarrow \infty$, $p = 2, 3$. Hence,

$$\mathbf{E}\{U_{jj}(t) - \widehat{U}_{jj}(t)\} = O(n^{-1/2}), \quad n \rightarrow \infty.$$

This and (3.130) yield (3.129) and finish the proof of (i).

(ii) The proof of (3.107) repeats with natural modifications the one of the first part of (i). Namely, similarly to (3.112) – (3.118) we have for $V_n = \mathbf{Var}\{v_n(t)\}$:

$$\begin{aligned} V_n &= \mathbf{E}\{[v_n(t) - \widehat{v}_n(t)]v_n^\circ(-t)\} \\ &= \frac{it}{2} \int_0^1 \left[\frac{1}{\sqrt{sn^3}} \sum_{i,k=1}^n \mathbf{E}\{W_{ik}^{(n)} \Phi_{ik}\} - \frac{1}{\sqrt{(1-s)n^3}} \sum_{i,k=1}^n \mathbf{E}\{\widehat{W}_{ik} \Phi_{ik}\} \right] ds \\ &= \frac{it}{2} \int_0^1 \left[\sum_{p=2}^4 T_p^{(n)} + \varepsilon_4 \right] s^{-1/2} ds, \end{aligned} \quad (3.131)$$

where now

$$\begin{aligned} T_p^{(n)} &= \frac{1}{p!n^{(3+p)/2}} \sum_{i,k=1}^n \kappa_{p+1,ik} \mathbf{E}\{D_{ik}^p \Phi_{ik}\}, \quad p = 2, 3, 4, \\ \Phi_{ik} &= U_k(t, s) v_n^\circ(-t), \quad |D_{ik}^l(s) \Phi_{ik}| \leq C(|t|), \end{aligned}$$

$C(|t|)$ is a polynomial in $|t|$ with positive coefficients, and

$$|\varepsilon_4| \leq \frac{C_4 w_6}{n^4} \sum_{i,k=1}^n \sup_{M \in \mathcal{S}_n} |D_{ik}^5 \Phi_{ik}| \leq C(|t|) n^{-2}. \quad (3.132)$$

Using the argument leading to (3.127) and (3.128) it can be shown that

$$T_p^{(n)} \leq C(|t|)(n^{-1}V_n^{1/2} + n^{-2}), \quad p = 2, 3, \quad T_4^{(n)} \leq C(|t|)n^{-2}.$$

This and (3.131) – (3.132) allow us to write the inequality

$$V_n \leq C(|t|)(n^{-1}V_n^{1/2} + n^{-2}),$$

valid for any real t and implying (3.107).

(iii) Statement (iii) was proved in Lemma 3.1 of [7].

(iv) Let $\bar{t} = (t_1, t_2)$ and

$$\widehat{v}_{n1}(\bar{t}) = n^{-1/2} \sum_{k=1}^n \widehat{U}_{jk}(t_1) \widehat{U}_{kk}(t_2)$$

with \widehat{U} of (3.111). We have similar to (3.115)

$$\begin{aligned} v_{n1}(\bar{t}) - \widehat{v}_{n1}(\bar{t}) &= \int_0^1 n^{-1/2} \frac{\partial}{\partial s} \sum_{k=1}^n U_{jk}(t_1, s) U_{kk}(t_2, s) ds \\ &= \frac{i}{2} \int_0^1 n^{-1/2} \sum_{p,q=1}^n (s^{-1/2} W_{pq}^{(n)} - (1-s)^{-1/2} \widehat{W}_{pq}) \Phi_{pq}(\bar{t}, s) ds, \end{aligned} \quad (3.133)$$

where now

$$\begin{aligned} \Phi_{pq}(\bar{t}, s) &= n^{-1/2} \sum_{k=1}^n (U_{jp} * U_{qk})(t_1, s) U_{kk}(t_2, s) + U_{jk}(t_1, s) (U_{kp} * U_{qk})(t_2, s), \\ |D_{pq}^i \Phi_{pq}(\bar{t}, s)| &\leq C(\bar{t}), \quad i \in \mathbb{N}. \end{aligned}$$

Applying (2.16) and (2.20) with $p = 3$ and $\Phi = \Phi_{pq}$, we get

$$\begin{aligned} \mathbf{E}\{v_{n1}(\bar{t}) - \widehat{v}_{n1}(\bar{t})\} &= \int_0^1 \left[\frac{\mu_3}{2n^{3/2}} \sum_{p,q=1}^n \mathbf{E}\{D_{pq}^2 \Phi_{pq}(\bar{t}, s)\} \right. \\ &\quad \left. + \frac{\kappa_4}{6n^2} \sum_{p,q=1}^n \mathbf{E}\{D_{pq}^3 \Phi_{pq}(\bar{t}, s)\} \right] s^{-1/2} ds + O(n^{-1/2}), \quad n \rightarrow \infty, \end{aligned} \quad (3.134)$$

where κ_4 is defined in (3.12). Using the argument leading to (3.127) and (3.128) and based on (2.10) – (2.12), it can be shown that both the terms in the square brackets of (3.134) are of the order $O(n^{-1/2})$, $n \rightarrow \infty$. Hence,

$$\mathbf{E}\{v_{n1}(\bar{t})\} - \mathbf{E}\{\widehat{v}_{n1}(\bar{t})\} = O(n^{-1/2}), \quad n \rightarrow \infty. \quad (3.135)$$

Moreover, replacing $\Phi_{pq}(\bar{t}, s)$ with $\Phi_{pq}(\bar{t}, s)v_{n1}^\circ(\bar{t})$ we can also obtain

$$\mathbf{Var}\{v_{n1}(\bar{t})\} = \mathbf{E}\{(v_{n1}(\bar{t}) - \widehat{v}_{n1}(\bar{t}))v_{n1}^\circ(\bar{t})\} = O(n^{-1/2}), \quad n \rightarrow \infty. \quad (3.136)$$

Now it follows from (3.135) – (3.136) that to finish the proof of (iii) it remains to show that

$$\lim_{n \rightarrow \infty} \mathbf{E}\{\widehat{v}_{n1}(\bar{t})\} = 0. \quad (3.137)$$

Indeed, since by the orthogonal invariance of the GOE probability measure we have

$$\mathbf{E}\{\widehat{U}_{jk}(t)\} = \delta_{jk} \mathbf{E}\{\widehat{v}_n(t)\},$$

where $\widehat{v}_n(t) = \text{Tr } \widehat{U}(t)$, $|\widehat{v}_n(t)| \leq 1$, then in view of (3.9) and (2.11)

$$\begin{aligned} \mathbf{E}\{\widehat{v}_{n1}(t_1, t_2)\} &= n^{-1/2} \mathbf{E}\{\widehat{v}_n(t_1)\} \mathbf{E}\{\widehat{v}_n(t_2)\} \\ &+ n^{-1/2} \sum_{k=1}^n \mathbf{E}\left\{\widehat{U}_{jk}(t_1) \widehat{U}_{kk}^\circ(t_2)\right\} = O(n^{-1/2}), \quad n \rightarrow \infty, \end{aligned} \quad (3.138)$$

and we get (iv).

(v) The scheme of the proof of (iv) is the same as the one of (iii). Namely, we have similar to (3.133) – (3.134) (see also (3.115)):

$$v_{n2}(\bar{t}) - \widehat{v}_{n2}(\bar{t}) = \frac{i}{2} \sum_{m=1}^l \int_0^1 \frac{1}{\sqrt{n}} \sum_{p,q=1}^n (s^{-1/2} W_{pq}^{(n)} - (1-s)^{-1/2} \widehat{W}_{pq}) \Phi_{pqm}(\bar{t}, s) ds,$$

where now

$$\begin{aligned} \Phi_{pqm}(\bar{t}, s) &= \sum_{k=1}^n (U_{jp} * U_{qk})(t_m, s) \prod_{m' \neq m} U_{jk}(t_{m'}, s), \\ |D_{pq}^l \Phi_{pqm}(\bar{t}, s)| &\leq C(\bar{t}), \quad l \in \mathbb{N}. \end{aligned}$$

Applying (2.16) and (2.20) with $p = 3$ one can get an analog of (3.134) and then show that (cf (3.135) – (3.136))

$$\mathbf{E}\{v_{n2}(\bar{t})\} - \mathbf{E}\{\widehat{v}_{n2}(\bar{t})\} = O(n^{-1/2}), \quad n \rightarrow \infty,$$

and

$$\mathbf{Var}\{v_{n2}(\bar{t})\} = O(n^{-1/2}), \quad n \rightarrow \infty.$$

So, it remains to prove that

$$\lim_{n \rightarrow \infty} \mathbf{E}\{\widehat{v}_{n1}(\bar{t})\} = \prod_{m=1}^l v(t_m). \quad (3.139)$$

Applying (2.16) and (2.12) and then (3.23) and (2.11), we get

$$\begin{aligned} \frac{\partial}{\partial t_1} \mathbf{E}\{\widehat{v}_{n2}(\bar{t})\} &= \frac{i}{\sqrt{n}} \sum_{k,p=1}^n \mathbf{E}\left\{\widehat{W}_{jp} \widehat{U}_{kp}(t_1) \prod_{m=2}^l \widehat{U}_{jk}(t_m)\right\} \\ &= -\frac{w^2}{n} \sum_{k=1}^n \mathbf{E}\left\{(t_1 \widehat{U}_{jk}(t_1) + (n \widehat{v}_n * \widehat{U}_{jk})(t_1)) \prod_{m=2}^l \widehat{U}_{jk}(t_m) + \sum_{m=2}^l \prod_{m' \neq m} \widehat{U}_{jk}(t_{m'}) \right. \\ &\quad \left. \times \int_0^{t_m} (\widehat{U}_{jj}(t_m - s_1) \widehat{U}_{kk}(t_1 + s_1) + \widehat{U}_{jk}(t_m - s_1) \widehat{U}_{jk}(t_1 + s_1)) ds_1\right\}, \end{aligned} \quad (3.140)$$

so that by (3.23) and (2.11) $\partial \mathbf{E}\{\widehat{v}_{n2}(\bar{t})\} / \partial t_1 = O(1)$, $n \rightarrow \infty$, and by the symmetry

$$\frac{\partial}{\partial t_m} \mathbf{E}\{\widehat{v}_{n2}(\bar{t})\} = O(1), \quad n \rightarrow \infty, \quad m = 1, \dots, l.$$

Hence, there exists a subsequence $\{\mathbf{E}\{\widehat{v}_{n_i2}(\bar{t})\}\}$ that converges uniformly on any compact set of \mathbb{R}^l . Now, applying the Duhamel formula (2.8) and then (3.140), we obtain

$$\begin{aligned} \mathbf{E}\{\widehat{v}_{n2}(\bar{t})\} &= \mathbf{E}\left\{\prod_{m=2}^l \widehat{U}_{jj}(t_m)\right\} + \int_0^{t_1} \frac{i}{\sqrt{n}} \sum_{k,p=1}^n \mathbf{E}\left\{\widehat{W}_{jp} \widehat{U}_{kp}(s) \prod_{m=2}^l \widehat{U}_{jk}(t_m)\right\} ds \\ &= \mathbf{E}\left\{\prod_{m=2}^l \widehat{U}_{jj}(t_m)\right\} - w^2 \int_0^{t_1} ds \int_0^s \mathbf{E}\{\widehat{v}_n(s - s_1)\} \mathbf{E}\{\widehat{v}_{n2}(s_1, t_2, \dots, t_l)\} ds_1 + r_n, \end{aligned}$$

where

$$r_n = -\frac{w^2}{n} \int_0^{t_1} \sum_{k=l}^n \mathbf{E} \left\{ (s \widehat{U}_{jk}(s) + (n \widehat{v}_n^\circ * \widehat{U}_{jk})(s)) \prod_{m=2}^l \widehat{U}_{jk}(t_m) \right. \\ \left. + \sum_{m=2}^l \prod_{m' \neq m} \widehat{U}_{jk}(t_{m'}) \int_0^{t_m} (\widehat{U}_{jj}(t_m - s_1) \widehat{U}_{kk}(s + s_1) + \widehat{U}_{jk}(t_m - s_1) \widehat{U}_{jk}(s + s_1)) ds_1 \right\} ds,$$

and by the Schwarz inequality, (2.11), and (3.107)

$$r_n = O(n^{-1}), \quad n \rightarrow \infty.$$

Taking into account (3.106) we obtain that every limit of converging subsequence

$$v_2(\bar{t}) = \lim_{i \rightarrow \infty} \widehat{v}_{n_i 2}(\bar{t})$$

satisfies the equation:

$$v_2(\bar{t}) + w^2 \int_0^{t_1} ds \int_0^s v(s - s_1) v_2(s, t_2, \dots, t_l) ds_1 = \prod_{m=2}^l v(t_m). \quad (3.141)$$

Applying the generalized Fourier transform with respect to the variable t_1 (see Proposition 2.1), we get

$$\widetilde{v}_2(z, t_2, \dots, t_l) (1 + w^2 z^{-1} \widetilde{v}(z)) = z^{-1} \prod_{m=2}^l v(t_m)$$

with \widetilde{v} of (3.41). Hence, $\widetilde{v}_2(z, t_2, \dots, t_l) = \widetilde{v}(z) \prod_{m=2}^l v(t_m)$, and

$$v_2(\bar{t}) = \prod_{m=1}^l v(t_m).$$

This completes the proof of (v) and the proof of the Lemma. ■

After this paper was completed we became aware of paper [11] by Soshnikov et al in which Theorem 3.4 was proved by another method and under weaker conditions.

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